How many zeroes?

Counting solutions of systems of polynomials via toric geometry at infinity

Pinaki Mondal

To my parents Purnima Mondal and Monojit Mondal

Preface

In this book we describe an approach through *toric geometry* to the following problem: "estimate the number (counted with appropriate multiplicity) of isolated solutions of n polynomial equations in n variables over an algebraically closed field $\mathbb{k}$." The outcome of this approach is the number of solutions for "generic" systems in terms of their *Newton polytopes*, and an explicit characterization of what makes a system "generic." The pioneering work in this field was done in the 1970s by Kushnirenko, Bernstein and Khovanskii, who completely solved the problem of counting solutions of generic systems on the "torus" $(\mathbb{k} \setminus \{0\})^n$. In the context of our problem, however, the natural domain of solutions is not the torus, but the affine space $\mathbb{k}^n$. There were a number of works on extending Bernstein's theorem to the case of affine space, and recently it has been completely resolved, the final steps having been carried out by the author.

The aim of this book is to present these results in a coherent way. We start from the beginning, namely Bernstein's beautiful theorem which expresses the number of solutions of generic systems on the torus in terms of the *mixed volume* of their Newton polytopes. We give complete proofs, over arbitrary algebraically closed fields, of Bernstein's theorem, its recent extension to the affine space, and some other related applications including generalizations of Kushnienko's results on *Milnor numbers* of hypersurface singularities which in 1970s served as a precursor to the development of toric geometry. Our proofs of all these results share several key ideas, and are accessible to someone equipped with the knowledge of basic algebraic geometry. This book can serve as a companion to introductory courses on algebraic geometry or toric varieties. While it does *not* provide a comprehensive introduction to algebraic geometry, it does develop the relevant parts of the subject from the beginning (modulo some explicitly stated basic results) with lots of examples and exercises, and can be used as a quick introduction to basic algebraic geometry. We hope the readers who take that undertaking will be rewarded by a deep understanding of the affine Bézout problem.

Acknowledgements. It was Pierre Milman who wanted me to write a book; it would not have been possible without his constant encouragement and support - with mathematics, and all sorts of things beyond mathematics - throughout these years. Even though the scope of the final version is considerably limited compared to his vision, I offer it as a first step. The encouragement from Eriko Hironaka worked as a catalyst during a critical period when the project was stuck. I sincerely thank Jan Stevens who sent numerous corrections after reading one of the earlier drafts. Najma Ahmad, Kinjal Dasbiswas, Naren Hoovinakatte, and especially, Jonathan Korman read parts of earlier drafts and gave important suggestions. Thanks are also due to the referees and editors, especially Keith Taylor, whose suggestions significantly improved the exposition. Over the last few years the work on this book took a great portion of my time owed to my friends and family, especially my mother Purnima Mondal and brother Protim Mondol. The application of points at infinity to Chickens' Road Crossing problem is due to Shatabdi Sarker; its presentation given in this book is due to Tanzil Rashid.

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CHAPTER I

Introduction

1. The problem and the results

This book is about the problem of computing the number of solutions of systems of polynomials, or equivalently, the number of points of intersection of the sets of zeroes of polynomials. In this section we formulate the precise version of the problem we are going to study and give an informal description of the results. One natural observation that simplifies the problem is that *intersection multiplicity* should be taken into account, e.g. even though a tangent line intersects a parabola at only one point, it should be counted with multiplicity two (see fig. 1).

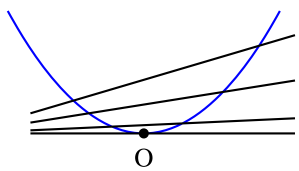


FIGURE 1. A tangent line intersects a parabola at a point with multiplicity two

The geometric intuition for intersection multiplicity is the "principle of continuity," the principle that continuous perturbations of systems result in continuous changes of associated metrics or invariants[1]. Since the number of points of intersection is a *discrete* invariant of a system, it follows that it must not change under continuous perturbation. However, over real numbers points of intersection may disappear upon an infinitesimal deformation (see fig. 2). On the other hand, this problem disappears if one also counts "imaginary" solutions (this is why the intersection theory over complex numbers, or, more generally, an algebraically closed field, is easier than the intersection theory over real numbers). In this book we will consider polynomial systems defined over algebraically closed fields[2].

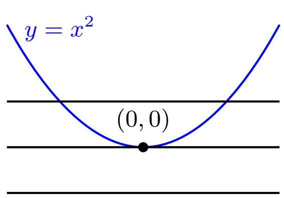


FIGURE 2. Disappearance of real points of intersection

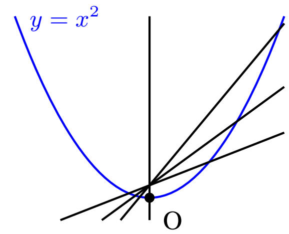
[1]"Consider an arbitrary figure in general position ... Is it not obvious that if ... one begins to change the initial figure by insensible steps, or applies to some parts of the figure an arbitrary continuous motion, then is it not obvious that the properties and relations established for the initial system remain applicable to subsequent states of this system provided that one is mindful of particular changes, when, say, certain magnitudes vanish, change direction or sign, and so on—changes which one can always anticipate a priori on the basis of reliable rules." – J. V. Poncelet, the foremost exponent of the principle of continuity, in the introduction of *Traité des propriétés projectives des figures* (1822), as cited in [**Ros05**].

[2]... which Poncelet probably would not have approved of, given his attitude towards consideration of complex solutions; see [**Gra11**, Section 4.2] for a most interesting account of this history.

If there are infinitely many solutions of a system of polynomials, then the solution set has positive dimensional components, and assigning multiplicity to these components is trickier; we bypass this problem in this book and consider only the number of *isolated*[3] solutions. This implies in particular we do not consider "underdetermined systems,"[4] since an underdetermined system over an algebraically closed field can only have either positive dimensional or empty sets of solutions. We also ignore "overdetermined systems"[4] because of the relative difficulty in assigning multiplicities. The final form of the subject of this book is thus the following:

Problem I.1 (Affine Bézout problem)**.** *Given n polynomials in n variables over an algebraically closed field $\mathbb{k}$, give a sharp estimate of the number of its isolated solutions counted with appropriate multiplicity, and determine the conditions under which it is exact.*

For $n = 1$, the fundamental theorem of algebra gives a complete answer: a polynomial of degree d has precisely d zeroes counted with multiplicity. For $n \geq 2$, there is a problem: points of intersection may run off to infinity (see fig. 3).



As secants approach the vertical line at O more and more closely, one of the points of intersection approaches O and the other goes to infinity.

FIGURE 3. A vertical line intersects the parabola at one point with multiplicity one

Any reasonable approach to problem I.1 therefore must take into account "intersections at infinity." A theorem named after E. Bézout (1730–1783) is the most basic result that does it satisfactorily.

THEOREM I.2 (Bézout's theorem, affine version). *The number of isolated solutions in $\mathbb{k}^n$ of n polynomials in n variables is at most the product of their degrees. Moreover, this bound is exact if and only if the only common solution of the* leading forms[5] *of the polynomials is the origin.*

Example I.3. Consider the system in fig. 3 consisting of the parabola $y - x^2 = 0$ and a line $ax + by + c = 0$. The Bézout bound is $2 \times 1 = 2$, and the leading forms are $-x^2$ and $ax + by$. As long as $b \neq 0$, the only solution to $-x^2 = ax + by = 0$ is $(0, 0)$, so that the bound is exact. However, if $b = 0$, i.e. the line is vertical, then any point of the form $(0, k)$, $k \in \mathbb{k}$, is a common solution of the leading forms. Consequently the Bézout bound overestimates the number of solutions in this case, as illustrated in fig. 3.

From the perspective of *projective geometry*, the Bézout bound is the number of intersections of polynomial hypersurfaces in the *projective space* $\mathbb{P}^n$, which is a *compactification* of the affine space $\mathbb{k}^n$ formed by adjoining a "hyperplane at infinity." Therefore the Bézout bound is exact if and only if the hypersurfaces do not intersect at any point at infinity on $\mathbb{P}^n$. However, as Gauss famously remarked,[6] infinity is the limit of some process, and curves which approach arbitrarily close to each other in one process may grow apart in another. A natural class of compactifications of $\mathbb{k}^n$ containing the projective space is that

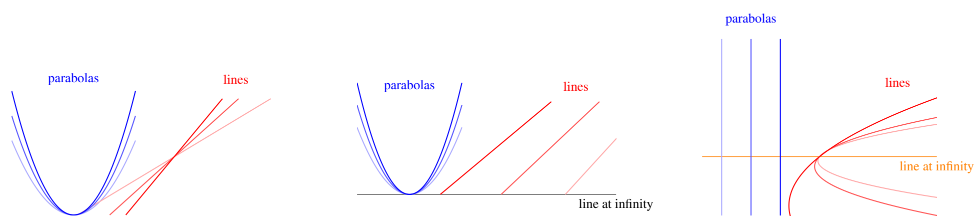
[3] A point is *isolated* in a set S if it is open in S.

[4] A system is *underdetermined* or *overdetermined* depending on whether the number of equations is smaller or greater than the number of variables.

[5] The *leading form* of a polynomial is the sum of its monomial terms with the highest degree; e.g. if $f = 2x^3 + 7x^2y - 9y^2 + 7xy - x + 1$, then its degree is 3 and the leading form is $2x^3 + 7x^2y$.

[6] Discussing his friend H. Schumacher's purported proof of the parallel postulate, Gauss wrote to him (as cited in [**Wat79**]), "I protest first of all against the use of an infinite quantity as a completed one, which is never permissible in mathematics. The infinite is only a façon de parler, where one is really speaking of limits to which certain ratios come as close as one likes while others are allowed to grow without restriction."

of *weighted projective spaces*. Given an n-tuple $\omega = (\omega_1, \dots, \omega_n)$ of positive integers, the corresponding *weighted rational curve* C^ω_a through a point $a = (a_1, \dots, a_n) \in \mathbb{k}^n$ is the curve parametrized by the map $t \mapsto (a_1 t^{\omega_1}, \dots, a_n t^{\omega_n})$. In the same way that in the projective space straight lines with different slopes are separated at infinity, in the weighted projective space $\mathbb{P}^n(1, \omega)$ the curves C^ω_a corresponding to distinct a are separated at infinity. See fig. 4 for an example with $\omega = (1, 2)$, in which case $\{C^\omega_a\}_a$ is the family of parabolas $\{{a_1}^2 y - a_2 x^2 = 0\}$. The "weight" of a monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ corresponding to ω is $\omega_1\alpha_1 + \cdots + \omega_n\alpha_n$. If f is a polynomial, then the corresponding *weighted degree* $\omega(f)$ of f is the maximum of the weights of all the monomials appearing in f. The *leading weighted homogeneous form* of f is the sum of all monomials (with respective coefficients) of f with the highest weight. Computing intersection numbers on $\mathbb{P}^n(1, \omega)$ leads to the "weighted Bézout theorem," of which the original theorem of Bézout (theorem I.2) is a special case (corresponding to $\omega = (1, \dots, 1)$).

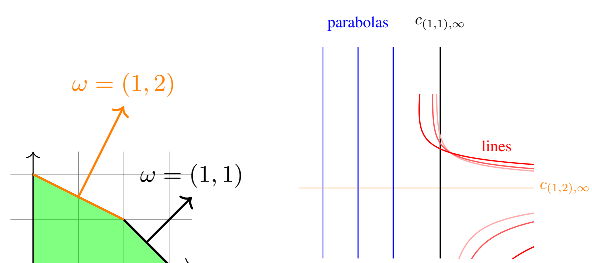


(A) $\mathbb{k}^2$ (B) $\mathbb{P}^2$ (C) $\mathbb{P}^2(1, 1, 2)$

FIGURE 4. $\mathbb{P}^2$ separates lines, but not parabolas, at infinity, whereas $\mathbb{P}^2(1, 1, 2)$ separates parabolas, but not lines, at infinity

THEOREM I.4 (Weighted Bézout theorem for positive weights). *Let ω be a weighted degree on the ring of polynomials with positive weights ω_i for x_i, $i = 1, \dots, n$. Then the number of isolated solutions of polynomials $f_1, \dots, f_n$ on $\mathbb{k}^n$ is bounded above by $(\prod_j \omega(f_j))/(\prod_j \omega_j)$. This bound is exact if and only if the leading weighted homogeneous forms of $f_1, \dots, f_n$ have no common solution other than the origin.*

Example I.5. Let $\omega = (1, 2)$, $f = y - x^2$ and $g = ax + c$, $a \neq 0$. Then $\omega(f) = 2$, $\omega(g) = 1$, and the leading weighted homogeneous forms of f and g are respectively $y - x^2$ and ax . The only solution to the leading weighted homogeneous forms of f and g with respect to ω is $(0, 0)$, so theorem I.4 implies that the number of solutions of $f = g = 0$ is precisely the weighted Bézout bound $(\omega(f)\omega(g)/(\omega(x)\omega(y)) = (2 \times 1)/(1 \times 2) = 1$, as we saw in fig. 3.



(A) $\mathcal{P}$ (B) A coordinate chart near infinity on $X_\mathcal{P}$

FIGURE 5. Parabolas and lines near curves at infinity on $X_\mathcal{P}$

The main class of compactifications considered in this book are *toric varieties* associated to *convex integral polytopes*[7]. If $\mathcal{P}$ is an n dimensional convex integral polytope in $\mathbb{R}^n$, then the outer normal to each

[7]A *convex integral polytope* in $\mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{R}^n$ with integer coordinates.

of its $(n-1)$-dimensional faces determines (up to a constant of proportionality) a weighted degree, and in the corresponding toric variety $X_{\mathcal{P}}$, weighted rational curves corresponding to each of these weights are separated. See fig. 5 for an example of a toric variety in which *both* parabolas and lines are separated at infinity. It has two curves at infinity (with respect to $\Bbbk^2$) corresponding to the two edges of $\mathcal{P}$ which are not along the axes; we denote these curves by $c_{\omega,\infty}$, where ω is the corresponding weight. Each $c_{\omega,\infty}$ separates the family of weighted rational curves corresponding to ω. Computing intersection numbers of hypersurfaces on toric varieties yields a beautiful result of D. Bernstein, which we now describe.

NP(f)
$f = 1 - x + 3x^3 + 4x^2y - 7y^2$

NP(g)
$g = 2 + x - y + xy + x^2y + xy^2$

FIGURE 6. Some Newton polytopes in dimension 2

The *Newton polytope* of a polynomial is the convex hull of all the exponents that appear in its expression, see fig. 6. V. I. Arnold noticed sometime in 1960s or 1970s that invariants of "generic" systems of polynomials tend not to depend on precise values of the coefficients of their monomials, but only on the combinatorial relations of the exponents of these monomials. The study of this phenomenon was a recurring topic at his seminars at Moscow University. While working on Arnold's question on determination of the *Milnor number*[8] at the origin of a generic polynomial, A. Kushnirenko discovered that if all polynomials have the same Newton polytope, then for generic systems the number of isolated solutions which do not belong to any coordinate hyperplane has a strikingly simple expression: it is simply $n!$ times the volume of this polytope! D. Bernstein soon figured out how to remove the restriction on Newton polytopes (about 130 years before this F. Minding [**Min41**] discovered a special case of Bernstein's theorem in dimension two[9]).

THEOREM I.6. *Let N be the number (counted with appropriate multiplicities) of the isolated zeroes of polynomials $f_1, \ldots, f_n$ on $(\Bbbk^*)^n := \Bbbk^n \setminus \bigcup_i \{x_i = 0\}$.*

(1) Kushnirenko [**Kou76**]: *If each f_j has the same Newton polytope $\mathcal{P}$, then $N \leq n!\operatorname{Vol}(\mathcal{P})$. If $\operatorname{Vol}(\mathcal{P})$ is nonzero, then the bound is exact if and only if the following condition holds:*

$$(*) \qquad \text{for each nontrivial weighted degree } \omega \text{, the corresponding leading forms of } f_1, \ldots, f_n \text{ do not have any common zero on } (\Bbbk^*)^n.$$

(2) Bernstein [**Ber75**]: *In general N is bounded above by the* mixed volume[10] *of the Newton polytopes of f_j. If the mixed volume is nonzero, then the bound is exact if and only if $(*)$ holds.*

Example I.7. If the Newton polytope of each polynomial contains the origin, then theorem I.6 in fact gives an upper bound on the number of isolated solutions on $\Bbbk^n$ and it is in general better than the bounds from theorems I.2 and I.4. For example, using the fact that mixed volume of two planar bodies $\mathcal{P}$ and $\mathcal{Q}$ is simply $\operatorname{Area}(\mathcal{P}+\mathcal{Q}) - \operatorname{Area}(\mathcal{P}) - \operatorname{Area}(\mathcal{Q})$ (example VII.3), we see that Bernstein's bound for the number of solutions of $f = g = 0$ (where f, g are as in fig. 6) is the area of the region shaded in blue in fig. 7, which is equal to 8. Bézout bound, on the other hand is $3 \times 3 = 9$; it is not hard to show that the 9 is also the best possible weighed Bézout bound.

[8]The *Milnor number* is an invariant of a singularity, see section XI.2.

[9]A. Khovanskii gives a summary of Minding's approach in [**BZ88**, Section 27.3]; an English translation of [**Min41**] by D. Cox and J. M. Rojas appears in [**GK03**].

[10]The *mixed volume* is the canonical multilinear extension (as a functional on convex bodies) of the volume to n-tuples of convex bodies in $\mathbb{R}^n$, see section VII.2 for a precise description.

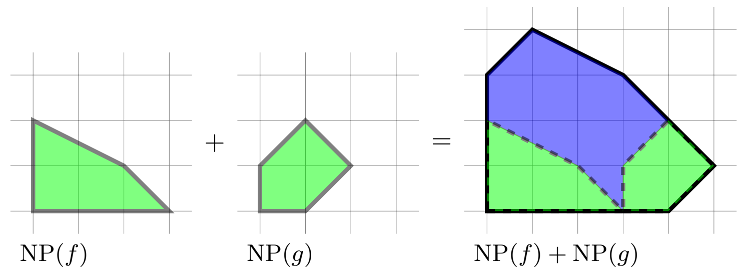


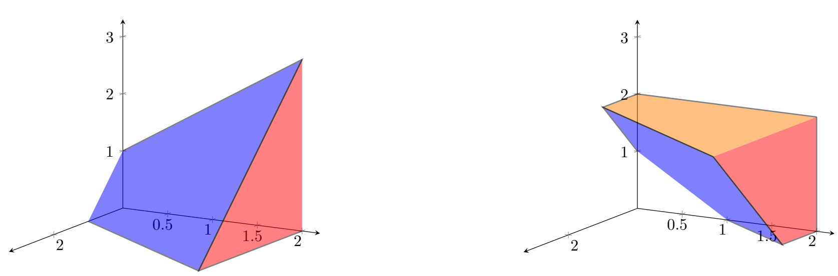
FIGURE 7. Minkowski sum of Newton polytopes of f and g

The natural domain of solutions of systems of polynomials over a field $\mathbb{k}$ is however not the torus $(\mathbb{k}^*)^n$, but the affine space $\mathbb{k}^n$. There are at least two different ways to extend Bernstein's formula to $\mathbb{k}^n$. The approach motivated by the *polynomial homotopy* method for solving polynomial systems is as follows: given polynomials $f_1, \ldots, f_n$, one starts with a deformed system $f_1 = c_1, \ldots, f_n = c_n$ with nonzero c_j. For generic $f_1, \ldots, f_n$ all solutions of the deformed system are in fact on the torus, and their number is given by Bernstein's theorem. Then one counts how many of these solutions approach isolated solutions of $f_1, \ldots, f_n$ as each $c_j \to 0$. This approach is taken in [**Kho78, HS95, LW96, RW96, Roj99**]. In particular, B. Huber and B. Sturmfels [**HS95**] found the general formula through this approach; however they proved it in a special case, and only in characteristic zero. J. M. Rojas [**Roj99**] observed that Huber and Sturmfels' formula works over all characteristics. The other approach is closer to Bernstein's original proof of his theorem: here one computes the number of "branches" of the curve defined by $f_2 = \cdots = f_n = 0$ and then the sum of the order of f_1 along these branches. General formulae through this approach were obtained by A. Khovanskii [unpublished][11] and the author [**Mon16**]. This formula requires knowing the *intersection multiplicity* at the origin of generic systems of polynomials. As an illustration we now state the weighted Bézout formula for weighted degrees with possibly *negative* weights[12]. Let ω be a weighted degree with nonzero weights $\omega_1, \ldots, \omega_n$. If $I_- := \{i : \omega_i < 0\}$, then the "general weighted Bézout bound" for the number of isolated zeroes of $f_1, \ldots, f_n$ is

$$\sum_{I \subseteq I_-} (-1)^{|I_-|-|I|} \frac{\prod_j \left(\max\{\omega(f_j), 0\} + \sum_{i \in I_-} |\omega_i| \deg_{x_i}(f_j)\right)}{\prod_i |\omega_i|} \tag{1}$$

(theorem X.36). Note that this reduces to the weighted Bézout bound from theorem I.4 in the case that each ω_i is positive, i.e. $I_- = \emptyset$. This bound is exact for generic $f_1, \ldots, f_n$, provided $\omega(f_j)$ is *nonnegative* for each j. In the general case, define

$$\mathcal{P}_\omega(f) := \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \alpha_i \geq 0 \text{ for each } i,\ \langle \omega, \alpha \rangle \leq \omega(f),\ \alpha_k \leq \deg_{x_k}(f) \text{ for each } j \in I_-\}$$



(A) $\omega = (1, -1, 1), \omega(f) = 1$ (B) $\omega = (1, -1, -1), \omega(f) = -1$

FIGURE 8. $\mathcal{P}_\omega(f)$ for $f = x_2^2 + x_2 x_3^2 + x_1 x_2 x_3$

[11] Khovanskii described his result to the author at the *Askoldfest* in 2017.

[12] For simplicity here we do not allow zero weights; see theorem X.36 for the statement without this restriction.

(see fig. 8). Given $I \subseteq \{1, \ldots, n\}$, let $\mathbb{R}^I$ be the $|I|$-dimensional coordinate subspace of $\mathbb{R}^n$ spanned by all x_i, $i \in I$. Then there is a collection $\mathscr{T}$ of subsets of $\{1, \ldots, n\}$ such that for each $I \in \mathscr{T}$, the number of distinct j such that $\mathcal{P}_\omega(f_j)$ touches $\mathbb{R}^I$ is precisely $|I|$, and the number of isolated zeroes of $f_1, \ldots, f_n$ is bounded by

$$(2) \qquad \sum_{I \in \mathscr{T}} \mathrm{MV}\left(\mathcal{P}_\omega(f_{j_1}) \cap \mathbb{R}^I, \ldots, \mathcal{P}_\omega(f_{j_{|I|}}) \cap \mathbb{R}^I\right) \times [\pi_{I'}(\mathcal{P}_\omega(f_{j'_1}), \ldots, \pi_{I'}(\mathcal{P}_\omega(f_{j'_{n-|I|}}))]_0$$

(see theorem X.38 for the precise statement), where

- $j_1, \ldots, j_{|I|}$ (respectively, $j'_1, \ldots, j'_{n-|I|}$) is the collection of indices j such that $\mathcal{P}_\omega(f_j)$ touches (respectively, does not touch) $\mathbb{R}^I$;
- $\mathrm{MV}(\cdot, \ldots, \cdot)$ is the mixed volume;
- $I' := \{1, \ldots, n\} \setminus I$ is the complement of I, and $\pi_{I'}$ is the natural projection onto the coordinate subspace of $\mathbb{R}^n$ spanned by all $x_{i'}$, $i' \in I'$, and
- $[\cdot, \ldots, \cdot]_0$ is the intersection multiplicity at the origin of systems of generic polynomials with given Newton polytopes.

The general formula for generic number of solutions on the affine space is no more difficult; it is of the same type as (2), i.e. it is a sum of products of mixed volumes and generic intersection multiplicities at the origin (see theorem X.4). However, to use it one needs to compute the generic intersection multiplicity at the origin (i.e. the second factor in the summands of (2)). In the special case that each polynomial is "convenient,"[13] a formula for generic intersection multiplicity was given by L. Ajzenberg and A. Yuzhakov [**AY83**]; a Bernstein-Kushnirenko type "non-degeneracy" condition, i.e. the condition for the bound being exact, was also known for convenient systems (see e.g. [**Est12**, Theorem 5]). In the general case Rojas [**Roj99**] gave a formula via Huber and Sturmfels' polynomial homotopy method. The non-degeneracy condition for the general case was established by the author in [**Mon16**].

As hinted above, the formula for the generic number of solutions on $\mathbb{k}^n$ is straightforward once one has the formula for generic intersection multiplicity at the origin. Sufficient criteria under which the bound is exact can also be obtained easily by adapting the Bernstein-Kushnirenko non-degeneracy condition ($*$); such criteria were given by several authors including Khovanskii [**Kho78**], Rojas [**Roj99**]. Precise non-degeneracy conditions, i.e. which are *necessary and sufficient* for the bound to be exact on $\mathbb{k}^n$, are however more subtle than ($*$); consider e.g. the problem of characterizing non-degenerate systems on $\mathbb{k}^3$ of the form

$$\begin{aligned} f_1 &= a_1 + b_1x_1x_2 + c_1x_2x_3 + d_1x_3x_1 \\ f_2 &= a_2 + b_2x_1x_2 + c_2x_2x_3 + d_2x_3x_1 \\ f_3 &= x_3(a_3 + b_3x_1x_2 + c_3x_2x_3 + d_3x_3x_1) \end{aligned}$$

where $a_j, b_j, c_j, d_j \in \mathbb{k}^*$ (this system is discussed in example X.16). If all a_j, b_j, c_j, d_j are generic, then it is straightforward to check directly that all common zeroes of f_1, f_2, f_3 on $\mathbb{k}^3$ are isolated and they appear on $(\mathbb{k}^*)^3$. Consequently, Bernstein's theorem implies that the number of solutions is the mixed volume of the Newton polytopes of the f_j, which equals 2. Now if $a_1 = a_2$, $b_1 = b_2$, and the remaining coefficients are generic, then ($*$) continues to be true, so that Bernstein's theorem applies and number of solutions on $(\mathbb{k}^*)^3$ is still 2; in particular, the system continues to be non-degenerate on $\mathbb{k}^3$. However, in this case the set of common zeroes of f_1, f_2, f_3 on $\mathbb{k}^3$ also has a *positive* dimensional component, namely the curve $\{x_3 = a_1 + b_1x_1x_2 = 0\}$. This situation never arises in the case of Bernstein's theorem; indeed, existence of a positive dimensional component makes a system violate ($*$) and its straightforward adaptations. Unlike the Bernstein-Kushnirenko non-degeneracy criterion, the correct non-degeneracy criterion for $\mathbb{k}^n$ needs to accommodate existence of positive dimensional components - it has to be able to differentiate between the cases when such a component leads to a loss of isolated solutions and when it does not; such a criterion was given by the author in [**Mon16**].

We mentioned above that the pioneering work of Kushnirenko on counting solutions of polynomial systems was motivated by his work on Milnor numbers of hypersurface singularities. In [**Kou76**] he gave a beautiful formula for a lower bound on the Milnor number, and showed that the bound is achieved by

[13] A polynomial or power series is *convenient* if for each j, there is $m_j \geq 0$ such that the coefficient of $x_j^{m_j}$ is nonzero.

Newton non-degenerate singularities if either the characteristic is zero or if the polynomial is convenient. It was however clear from the beginning that Newton non-degeneracy is not necessary for the formula to hold, and it also does not imply "finite determinacy."[14] C. T. C. Wall [**Wal99**] introduced another notion of non-degeneracy which implies finite determinacy and which also guarantees that the Milnor number can be computed by Kushnirenko's formula. S. Brzostowski and G. Oleksik [**BO16**] found the combinatorial condition which under Newton non-degeneracy is equivalent to finite determinacy. The Milnor number of a hypersurface at the origin is same as the intersection multiplicity at the origin of the partial derivatives of the defining polynomial (or power series). The non-degeneracy condition for intersection multiplicity therefore gives a natural starting point to study Milnor numbers. This condition generalizes both Newton non-degeneracy (for isolated singularities) and Wall's non-degeneracy condition; the author showed in [**Mon16**] that in positive characteristic this condition is sufficient, and in zero characteristic it is both necessary and sufficient, for the Milnor number to be generic.

The purpose of this book is to give a unified exposition of the results described above. In addition to Bernstein's theorem (over arbitrary algebraically closed fields), classical results proved in this book include weighted homogeneous and multi-homogeneous versions of Bézout's theorem; complete proofs (or even, statements) of these results are otherwise hard to find. We followed Bernstein's original proof for establishing the non-degeneracy conditions of his theorem; in particular we present his simple and ingenious trick to construct a curve of solutions that runs off to infinity in the case that the non-degeneracy condition ($*$) is not satisfied[15]. This book is the first part of a series of works on a constructive approach to compactifications of affine varieties started in the author's PhD thesis [**Mon10**], for which the affine Bézout problem served as a motivation. Based on the results of this book, in the next part we give a solution to the general version of the affine Bézout problem, i.e. give a recipe to compute the precise number (counted with multiplicity) of solutions of any given system of n polynomials in n variables. The algorithm is inductive; it consists of finitely many steps, and at each step a non-degeneracy criterion determines if the correct number has been computed. The estimate and non-degeneracy criterion for the number of solutions on $\mathbb{k}^n$ from chapter X of this book serve as the initial step of that algorithm.

2. Prerequisites

We tried to ensure that this book is accessible to someone with the mathematical maturity and algebra background of a second year mathematics graduate student. In the ideal case a reader would be familiar with the properties of algebraic varieties discussed in chapter III, so that (s)he could start with toric varieties in part 2 and only refer to results from part 1 if necessary. However, part 1 is self contained (modulo the dependencies explicitly stated in appendix A and section IV.3.1 and some commutative algebra results stated in appendices B and C) - with proper guidance it can be used as the material for a first course in algebraic geometry. One possible strategy for such a course would be to cover the chapters on algebraic varieties (chapter III), toric varieties (chapter VI), Bernstein-Kushnirenko theorem (chapter VII) and (weighted) Bézout's theorem (chapter VIII). The chapters on intersection multiplicity (chapter IV) and polytopes (chapter V) are included for completion - in a first course the required results from these chapters can simply be explained, perhaps via examples and/or pictures, instead of working out the details of the proofs. In particular, the proofs (and exercises) given in chapter V (polytopes) are elementary and a student should not have much difficulty in following them. The most sophisticated part of chapter IV (intersection multiplicity) is the concept of a "closed subscheme" of a variety and the fact that it can be locally defined by ideals determined by regular functions; the other results are basic facts about intersection multiplicity of n regular functions at a nonsingular point a of an n dimensional variety (e.g. that they can be defined via the "order" at a of one of the functions along the curve defined by the other functions) and relevant properties of the "order" function at a point on a (possibly non-reduced) curve. While the proofs use somewhat

[14] i.e. it does not ensure that the singularity at the origin is isolated.

[15] The bound from Bernstein's theorem and the sufficiency of ($*$) for the bound can be established without much difficulty (and in a very elegant way) using the general machinery of intersection theory (see e.g. [**Ful93**, Section 5.4]). However, we do not know of any proof of the *necessity* of ($*$) using this approach which does not involve an adaptation of Bernstein's trick; in all probability it would be much more difficult otherwise, since establishing positivity of excess intersections is in general a hard problem. Bernstein's trick is a nontrivial example of an elementary argument faring better than a formidable machinery.

complicated algebra, the statements are intuitive, at least if one has some familiarity with basic properties of (complex) analytic functions.

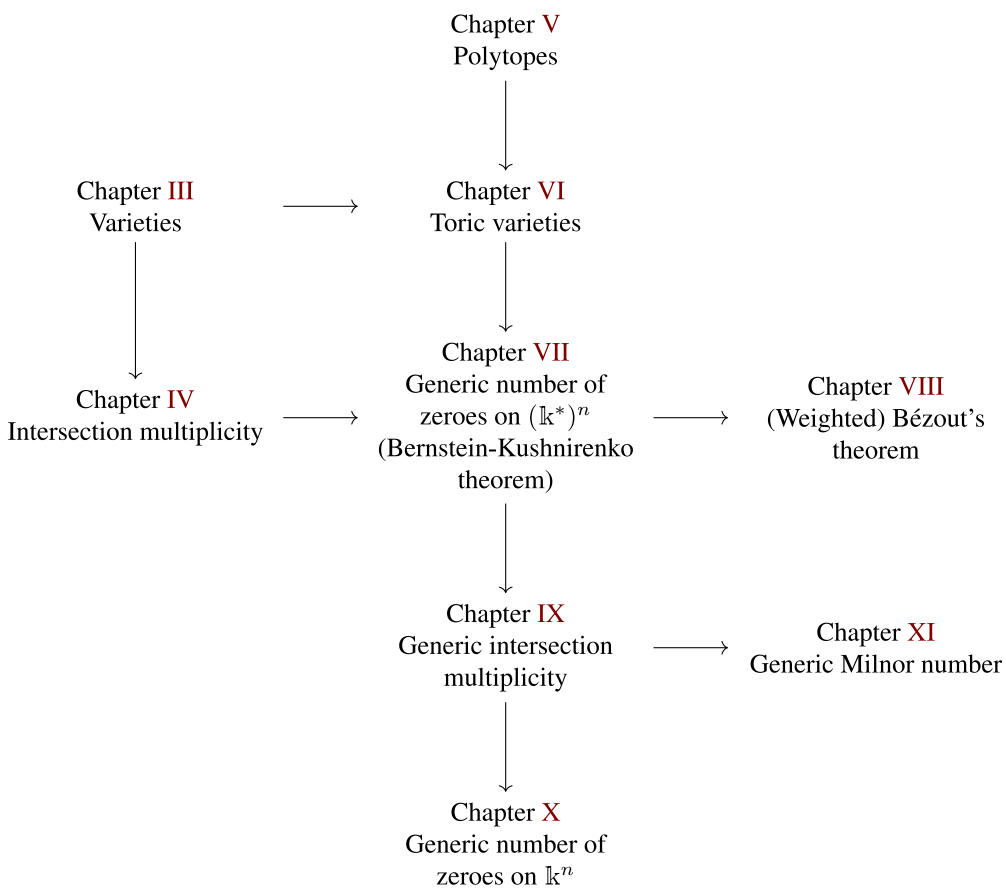
3. Organization

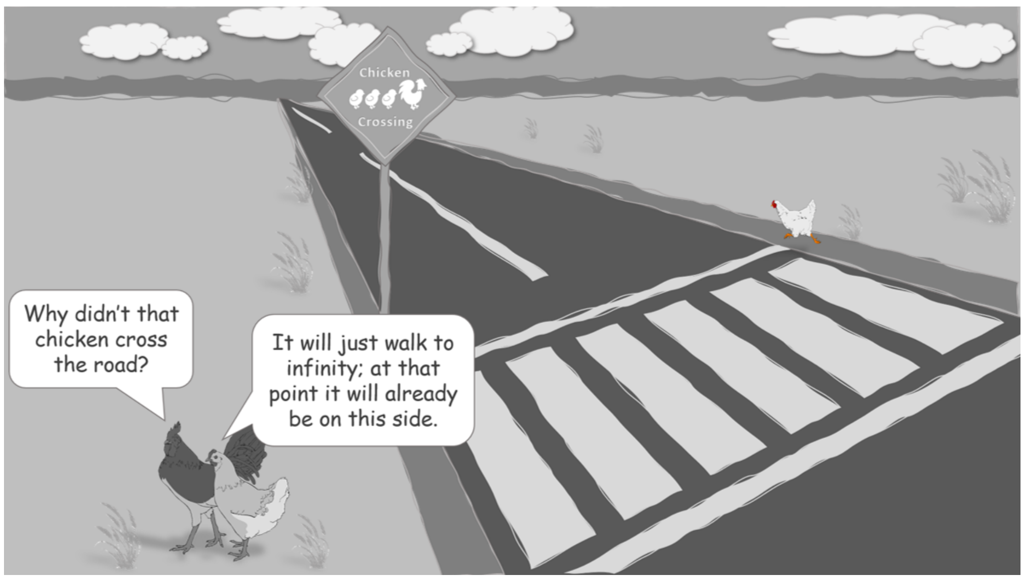
Part 1 and the first chapter of part 2 have been designed as parts of a textbook, with many exercises and examples. The goal was to develop efficiently (and in an elementary way) the theory needed to prove the results in the subsequent chapters. These latter chapters are more like those of a monograph; there are no exercises, but they do contain a number of examples. We now give a short description of each chapter.

In chapter III we develop the required theory of algebraic varieties. We tried to stress the geometric point of view where possible. A number of results have been developed through exercises; ample hints have been provided to ensure that no single step of any exercise is very difficult. In chapter IV we describe basic properties of intersection multiplicity (of n regular functions at a nonsingular point of an n dimensional variety), in particular how it can be computed using curves. After giving simple examples to illustrate that a satisfactory treatment of intersection multiplicity would need to incorporate non-reduced rings, we give a short introduction to "closed subschemes of a variety"[16]. A number of examples presented in chapters III and IV were taken from answers to the question *Algebraic geometry examples* [**hba**] posed by R. Borcherds on *MathOverflow*. Chapter V is a compilation (with complete proof) of the properties of convex polyhedra which, together with the results of chapter III, constitute the foundation on which we introduce toric varieties in chapter VI. In chapter VI we mainly discuss those properties of toric varieties which are required for the results in the subsequent chapters. In chapter VII we prove Bernstein's theorem and present some of its basic applications to convex geometry. In chapter VIII we apply Bernstein's theorem to prove the weighted homogeneous and multi-homogeneous versions of Bézout's theorem. Chapter IX contains the results on the generic bound and non-degeneracy conditions for intersection multiplicity at the origin, which we use in chapter X to compute the generic bounds and non-degeneracy conditions for the number of solutions of polynomial systems on $\mathbb{k}^n$. It turns out that one can as easily replace $\mathbb{k}^n$ by an arbitrary Zariski open subset of $\mathbb{k}^n$ - the results of chapter X are derived in this greater generality. In chapter X we also use the main results to derive generalizations of weighted homogeneous and multi-homogeneous versions of Bézout's theorem applicable to weighted degrees with possibly zero or negative weights. In chapter XI we apply the results from chapter IX to the study of Milnor numbers; in particular, we derive and generalize classical results of Kushnirenko on Milnor numbers. Chapters VII, X and XI end with selections of open problems (mostly combinatorial in nature).

[16]We decided to omit definitions of general sheaves and schemes since we do not use these notions anywhere in this book. On the other hand, once one really understands the special cases of "sheaves of ideals" and "closed subschemes of a variety," which are discussed in chapter IV, the leap to the general notions will be natural.

Chapter Dependencies





On the projective space chickens have more than one way of crossing roads