Towards a Bezout-type theory of affine varieties

by

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Abstract

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We study projective completions of affine algebraic varieties (defined over an algebraically closed field $\mathbb{K}$) which are given by filtrations, or equivalently, integer valued 'degree like functions' on their rings of regular functions. For a polynomial map $P := (P_1, \ldots, P_n) : X \to \mathbb{K}^n$ of affine varieties with generically finite fibers, we prove that there are completions of the source such that the intersection of completions of the hypersurfaces $\{P_j = a_j\}$ for generic $(a_1, \ldots, a_n) \in \mathbb{K}^n$ coincides with the respective fiber (in short, the completions 'do not add points at infinity' for P). Moreover, we show that there are 'finite type' completions with the latter property, i.e. determined by the maximum of a finite number of 'semidegrees', by which we mean degree like functions that send products into sums. We characterize the latter type completions as the ones for which ideal I of the 'hypersurface at infinity' is radical. Moreover, we establish a one-to-one correspondence between the collection of minimal associated primes of I and the unique minimal collection of semidegrees needed to define the corresponding degree like function. We also prove an 'affine Bezout type' theorem for polynomial maps P with finite fibers that admit semidegrees corresponding to completions that do not add points at infinity for P. For a wide class of semidegrees of a 'constructive nature' our Bezout-type bound is explicit and sharp.

Dedication

To my father,

who would have been very happy.

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Chapter 0

Introduction

0.1 Introduction

The most prominent example of complete spaces in algebraic geometry is perhaps the complex projective space $\mathbb{P}^n(\mathbb{C})$. The usual way of constructing $\mathbb{P}^n(\mathbb{C})$ is to view it as the space of lines in $\mathbb{C}^{n+1}$ that pass through the origin. Equivalently, via Hilbert's Nullstellensatz, it can be identified with the *maximal homogeneous ideals* of the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$ that do not contain the *irrelevant* ideal generated by $x_0, \ldots, x_n$. In algebraic geometry jargon, the preceding sentence states that $\mathbb{P}^n(\mathbb{C})$ is the *Proj* of the graded $\mathbb{C}$-algebra $\mathbb{C}[x_0, \ldots, x_n]$, which is graded by degree of polynomials. $\mathbb{C}$-algebra $\mathbb{C}[x_0, \ldots, x_n]$ is, on the other hand, isomorphic to the graded $\mathbb{C}$-algebra which is the direct sum of vector subspaces of polynomials in $(x_1, \ldots, x_n)$ of degree less than or equal to d over all non-negative integers d. Identifying polynomials in $(x_1, \ldots, x_n)$ with regular functions on $\mathbb{C}^n$, this gives us an algebraic way of constructing $\mathbb{P}^n(\mathbb{C})$ as a *completion* of $\mathbb{C}^n$. It is well known that this construction can be generalized: let X be an arbitrary affine variety and $\mathcal{F} = \{F_d : d \geq 0\}$ be a *filtration* on the coordinate ring $\mathbb{C}[X]$ of X (which in our case means $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ is a sequence of vector subspaces of $\mathbb{C}[X]$ such that $\mathbb{C}[X] = \bigcup_{d \geq 0} F_d$, and $F_d F_e \subseteq F_{d+e}$). Then $\operatorname{Proj} \bigoplus_{d \geq 0} F_d$ is a completion of X, i.e. a complete variety which contains X as a dense open subset.

Giving a filtration $\mathcal{F}$ on $\mathbb{C}[X]$, on the other hand, is equivalent to defining a *degree like function* $\delta : \mathbb{C}[X] \to \mathbb{Z}$ which satisfies the following properties:

1. $\delta(f+g) \leq \max\{\delta(f), \delta(g)\}$ for all $f, g \in \mathbb{C}[X]$, with $<$ in the preceding equation implying $\delta(f) = \delta(g)$.
2. $\delta(fg) \leq \delta(f) + \delta(g)$ for all $f, g \in \mathbb{C}[X]$.

The vector spaces $F_d := \{f \in \mathbb{C}[X] : \delta(f) \leq d\}$ define a filtration on $\mathbb{C}[X]$ associated with δ and from that filtration, one constructs a completion of X as in the first paragraph. For $X = \mathbb{C}^n$ and δ equal to the usual degree of polynomials, the completion of $\mathbb{C}^n$ we get via this construction is $\mathbb{P}^n(\mathbb{C})$. If we take δ to be a more general *weighted degree*, we end up with the corresponding *weighted projective space*.

Both the usual and weighted degrees satisfy property 2 with exact *equality* instead of the inequality. We call the degree like functions which have this property *semidegrees*. In the theory of *toric geometry*, one associates a normal n-dimensional projective variety to a convex integral polytope of dimension n. Each facet (i.e. codimension one face) of such a polytope $\mathcal{P}$ defines a weighted degree on $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. It turns out (cf. Example 2.1.3) that the toric variety $X_{\mathcal{P}}$ associated to $\mathcal{P}$ is precisely the completion of the n-torus $(\mathbb{C}^*)^n$ corresponding to the degree like function $\delta_{\mathcal{P}}$ which is the maximum of the weighted degrees corresponding to the facets of $\mathcal{P}$. One of the guiding principle of this thesis is the conviction that the completion of $\mathbb{C}^n$ coming from a semidegree should 'behave in the same way' as the weighted projective spaces, which are completions of $\mathbb{C}^n$ corresponding to weighted degrees. As the analogue of the $\delta_{\mathcal{P}}$ corresponding to a polytope $\mathcal{P}$, the same principle leads us to consider *subdegrees*, i.e. degree like functions which are the maximum of finitely many semidegrees. As a manifestation of this principle we prove that in the completion $\bar{X}$ of an arbitrary affine variety X corresponding to a subdegree δ on $\mathbb{C}[X]$, the irreducible components of the *hypersurface at infinity* $X_\infty := \bar{X} \setminus X$ have a canonical one-to-one correspondence with the semidegrees associated to δ, in the same way that the irreducible components of $X_{\mathcal{P}} \setminus (\mathbb{C}^*)^n$ correspond to the facets of $\mathcal{P}$ (see Theorem 2.2.1 and Proposition 2.2.12). Given a completion $X \hookrightarrow Z$ determined by a degree like function δ, we introduce a 'normalization' procedure to produce a $\bar{\bar{\delta}}$ which is a subdegree (Theorem 2.2.16). When X is normal, the corresponding completion $\bar{Z}$ is the normalization of Z. The construction of $\bar{\delta}$ from δ generalizes the well known procedure of constructing the normalization of a non-normal toric variety (determined by a finite subset of a lattice) by 'filling the holes' [6, Theorem 3.A.5].

This work started as a project to understand 'affine Bezout type' theorems. By affine Bezout type theorems we mean those which estimate the number of solutions of a system of equations on an *affine* variety (as opposed to the usual version of Bezout theorem, which gives a formula for the number of solutions of n polynomials on the *projective* space $\mathbb{P}^n(\mathbb{C})$). The most famous theorems of this kind, apart from the Bezout theorem, are perhaps those of Kushnirenko [20] and Bernstein [2]. Bernstein's theorem is more general, and it states the following: let $\Delta_1, \ldots, \Delta_n$ be convex polytopes in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$. Then for generic Laurent polynomials $f_1, \ldots, f_n$ such that the *Newton polytope* of f_i lies inside Δ_i, the number of solutions in $(\mathbb{C}^*)^n$ of the system $f_1(x) = \cdots = f_n(x) = 0$ is equal to $n!$ times the *mixed volume* of $\Delta_1, \ldots, \Delta_n$. There have been numerous works which generalize this theorem and provide formulae for the number of solutions of systems of equations on affine varieties, the systems being *generic* in some suitable sense (see, e.g. [15], [16], [21], [31], [30], [32]). Curiously, most of these constructions have one property in common - they involve some polynomial map $f := (f_1, \ldots, f_q) : X \to \mathbb{C}^q$ of affine varieties with finite fibers and completions $X \hookrightarrow Z$ of X which satisfy the following property:

$$\text{for generic } a := (a_1, \ldots, a_q) \in \mathbb{C}^q,\ \bar{H}_1(a) \cap \cdots \cap \bar{H}_q(a) \cap (Z \setminus X) = \emptyset, \tag{$*$}$$

where $H_i(a) := \{x \in X : f_i(x) = a_i\}$ and $\bar{H}_i(a)$ is the closure of $H_i(a)$ in Z for all $1 \leq i \leq q$. This observation suggests the following

Definition 0.1.1.

- A completion $\psi : X \hookrightarrow Z$ *preserves the intersection of subvarieties* $V_1, \ldots, V_k$ *of* X *at* ∞ if $\bar{V}_1 \cap \cdots \cap \bar{V}_k \cap X_\infty = \emptyset$, where $X_\infty := Z \setminus X$ is the set of 'points at infinity' and $\bar{V}_j$ is the closure of V_j in Z for every j.
- Given a collection $\bar{f} := \{f_i : 1 \leq i \leq q\}$ of polynomials on X, $a := (a_1, \ldots, a_q) \in \mathbb{C}^q$ and a completion ψ of X, we say that ψ *preserves* $\bar{f}$ *over* a *at* ∞ if ψ preserves the intersection of the hypersurfaces $H_i(a) := \{x \in X : f_i(x) = a_i\}$, $i = 1, \ldots, q$; we simply say completion ψ *preserves* $\bar{f}$ *at* ∞ if ψ preserves $\bar{f}$ over a at ∞ for a in a Zariski open subset of $\mathbb{C}^q$.

Remark - definition 0.1.2. Our results throughout this work remain valid with the property of preservation of a collection $\bar{f}$ at ∞ by completion ψ being replaced by a stronger property as follows:

- If for any linear coordinate change $(x_1, \ldots, x_q) \mapsto (\sum_{i=1}^q c_{1i}x_i, \ldots, \sum_{i=1}^q c_{qi}x_i)$ on $\mathbb{C}^q$, completion ψ preserves $\{\sum_{i=1}^q c_{ji}f_i\}_{1\leq j\leq q}$ over points $(\sum_{i=1}^q c_{1i}a_i, \ldots, \sum_{i=1}^q c_{qi}a_i)$ at ∞ (for a fixed point $a \in \mathbb{C}^q$), we say that completion ψ *preserves map* $f := (f_1, \ldots, f_q) : X \to \mathbb{C}^q$ *over* a *at* ∞, and simply say completion ψ *preserves map* f *at* ∞ if ψ preserves map f over a at ∞ for a in a Zariski open subset of $\mathbb{C}^q$.

In this thesis we prove that given any polynomial map $f : X \to \mathbb{C}^n$ with generically finite fibers, there exists a degree like function δ on the coordinate ring of X such that δ is a subdegree and the corresponding completion of X preserves map f at ∞ (Corollary 2.2.19). When δ is itself a semidegree, we derive an affine Bezout type formula for the size of generic fibers of f in terms of degree D of a d-uple completion of the resulting variety (Theorem 3.1.1). Using recent work of Kaveh and Khovanskii [17], we provide a description of D in terms of the volume of a convex body determined by δ (Proposition 3.1.5). For $X = \mathbb{C}^n$, we also describe an iterative procedure (starting with a weighted degree) of a construction of a semidegree and provide a simple algebraic formula for D (Theorem 3.2.7). Finally, we consider more general subdegrees δ and derive an upper bound for the size of generic fibers of f in terms of the degree of an appropriate completion of X we construct from δ (Theorem 3.3.2).

I express my gratitude to my advisor Professor Pierre Milman for posing the questions and guiding me throughout this work. I would also like to thank Professor Khovanskii for helpful suggestions, e.g. he pointed out the possibility of a connection between semidegrees and 'orders of poles at infinity' [18] and indicated the idea of the proof of theorem 1.3.5.

0.2 Historical Background

The history of Bezout's theorem can be traced at least as far back as 1680. Newton had observed by that time that the abscissas of the intersection points of two algebraic plane curves of degrees m and n are given by a polynomial equation of degree less than or equal to mn [29]. In 1720 Maclaurin's book 'Geometrica Organica' contained the theorem that two plane algebraic curves of degree m and n intersect at at most mn points, unless they have infinitely many points in common [22]. But it seems that a correct proof of the theorem was missing until Étienne Bézout came up with one in 1764, and he also generalized it to an arbitrary dimension $n \geq 2$ ([3], [4], [5]). There was a lot of work on this theorem in the 20th century. One major stream was about understanding the concept of multiplicity of intersections and extending it to different algebraic settings. We mention [35] as an example of a work in this direction. It also has an excellent survey - the little bit of history contained in this paragraph was taken from there.

On the other hand, our motivation was to understand the developments in "affine Bezout type" theorems. The statement of Bezout theorem when viewed from this perspective is as follows:

Theorem (Bezout). *Let $f := (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that*

$\#f^{-1}(0)$ is finite. then

$$\#f^{-1}(0) \leq \prod_{i=1}^{n} \deg(f_i), \tag{1}$$

where $\#f^{-1}(0)$ is the size of $f^{-1}(0)$ counted with multiplicity. (1) *holds with equality iff the following condition holds:*

> *the leading homogeneous components of f_i have only one common zero, namely the origin.* (E1)

There is also a weighted version of Bezout's theorem which is more or less well known (see, e.g. [7]).

Theorem (Weighted Bezout)**.** *Let $f := (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $\#f^{-1}(0)$ is finite. Let δ be a* weighted degree *on $\mathbb{C}[x_1, \ldots, x_n]$ that assigns weight d_i to x_i. Then*

$$\#f^{-1}(0) \leq \frac{\prod_{i=1}^{n} \delta(f_i)}{\prod_{i=1}^{n} d_i}, \tag{2}$$

where $\#f^{-1}(0)$ is the size of $f^{-1}(0)$ counted with multiplicity. (2) *holds with equality iff the following condition holds:*

> *leading weighted homogeneous components of f_i have only one common zero, namely the origin.* (E2)

In mid 1970's Kushnirenko discovered a remarakable formula for the number of solutions of n *Laurent polynomials* (i.e. elements of $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$) on the torus $(\mathbb{C}^*)^n$ in terms of the volume of the *Newton polytope* $\mathcal{NP}(f_i)$ of the f_i's. Recall that the Newton polytope of an element $g := \sum a_\alpha x^\alpha \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ is the convex hull (in $\mathbb{R}^n$) of all α such that $a_\alpha \neq 0$.

Theorem (Kushnirenko [20])**.** *Let A be a finite subset of $\mathbb{Z}^n$ and $\mathcal{P}$ be the convex hull of A in $\mathbb{R}^n$. Let $f_i := \sum a_{i,\alpha} x^\alpha \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ such that $\mathcal{NP}(f_i) \subseteq \mathcal{P}$ for $i = 1, \ldots, n$. Then*

$$\#f^{-1}(0) \leq n! \operatorname{Vol}(\mathcal{P}), \tag{3}$$

where $\#f^{-1}(0)$ *is the number of the* isolated roots *of* $f_1, \ldots, f_n$ *on* $(\mathbb{C}^*)^n$ *counted with multiplicity, and* $\mathrm{Vol}(\mathcal{P})$ *is the* n*-dimensional Euclidean volume of* $\mathcal{P}$*.* (3) *holds with equality iff the following condition holds:*

$$\text{for all } \alpha \in \mathbb{Z}^n,\ f_{1,\alpha}, \ldots, f_{n,\alpha} \text{ have no common solutions in } (\mathbb{C}^*)^n, \tag{E3}$$

where $f_{i,\alpha}$ *is the* weighted leading form *of* f_i *corresponding to the weight* α*. In other words,* $f_{i,\alpha} := \sum_{\beta \in A_\alpha} a_{i,\beta} x^\beta$*, where* $A_\alpha := \{\beta \in A : \langle \alpha, \beta \rangle \geq \langle \alpha, \gamma \rangle \text{ for all } \gamma \in A\}$*.*

The observation crucial to our work is that conditions (E1) and (E2) are equivalent to the condition that an appropriate completion of $\mathbb{C}^n$ *preserves* $\{f_1, \ldots, f_n\}$ *at* ∞ *over the origin*, the completion being the usual projective space $\mathbb{P}^n(\mathbb{C})$ in the case of Bezout's theorem and the weighted projective space $\mathbb{P}^n(\mathbb{C}; 1, d_1, \ldots, d_n)$ in the case of weighted Bezout theorem. Similarly, when $\dim(\mathcal{P}) = n$, condition (E3) is equivalent to the condition that the *toric completion* $X_{\mathcal{P}}$ of $(\mathbb{C}^*)^n$ preserves $\{f_1, \ldots, f_n\}$ at ∞ over 0 (when $\dim \mathcal{P} < n$, variety $X_{\mathcal{P}}$ is *not* a completion of $(\mathbb{C}^*)^n$, but of $(\mathbb{C}^*)^m$ for some $m < n$; nonetheless there is a map $\phi_{\mathcal{P}} : (\mathbb{C}^*)^n \to X_{\mathcal{P}}$ and condition (E3) is equivalent to having empty intersection at ∞ of the closure in $X_{\mathcal{P}}$ of the image of $V(f_i)$ under $\phi_{\mathcal{P}}$). We refer the reader to section 2.0.1 for a brief introduction to toric varieties.

Shortly after Kushnirenko proved his theorem, Bernstein extended it to a more general setting in which he fixed n convex integral polytopes $\mathcal{P}_1, \ldots, \mathcal{P}_n$ and concluded that for all $f_1, \ldots, f_n$ with $\mathcal{NP}(f_i) \subseteq \mathcal{P}_i$, the number of isolated roots of $f_1, \ldots, f_n$ on $(\mathbb{C}^*)^n$ is bounded by $n!$ times the *mixed volume* of $\mathcal{P}_1, \ldots, \mathcal{P}_n$. He also shows that his estimate is exact for generic $f_1, \ldots, f_n$ satisfying $\mathcal{NP}(f_i) \subseteq \mathcal{P}_i$ for each i. After a far reaching generalization by A. Khovanskii, e.g. as in monograph [19], this estimate is now known as the "BKK" bound (where BKK stands for Bernstein, Kushnirenko and Khovanskii). The BKK bound was later refined by a number of authors including Huber and Sturmfels ([15], [16]), Li and Wang [21], Rojas ([31], [30], [32]). But as it happens with the three theorems stated in this section, the estimates we get in most of these works for the size of a fiber of a map are exact if and only if an appropriate completion preserves $\{f_1, \ldots, f_n\}$ at ∞ over the corresponding point. This is what motivated us to study the phenomenon of the preservation of the component hypersurfaces of a map at ∞, which resulted in this thesis.

In the appendix, together with a precise statement of Bernstein's theorem, we provide a proof of our interpretation of Kushnirenko-Bernstein's non-degeneracy condition (E3) as

$$\text{the completion } (\mathbb{C}^*)^n \hookrightarrow X_{\mathcal{P}} \text{ preserves } \{f_1, \ldots, f_n\} \text{ at } \infty \text{ over } 0, \tag{E}$$

where $X_{\mathcal{P}}$ is the toric variety associated with the *Minkowski sum* $\mathcal{P}$ of the Newton polytopes of components of f.

0.3 Organization

Throughout this thesis we work over an arbitrary algebraically closed field $\mathbb{K}$. In chapter 1 we set up the basic theory of the completions that we study in the rest of the thesis. In section 1.1 we define the notion of a filtration and give a characterization of the completions of an affine algebraic variety over $\mathbb{K}$ which come from filtrations on its coordinate ring. In section 1.2 we give several basic examples of filtrations and corresponding completions. At the end of this section we also give an example of a projective completion of an affine variety (which is a plane cubic minus a non-torsion point) which does not come from any filtration. On the other hand, we ask the following

Question. Does $\mathbb{K}^n$ admit a projective completion not induced by a filtration?

Section 1.3 is where we consider the question: "given a collection of subvarieties of an affine variety X with finite intersections, when does a completion of X preserve their intersection at ∞?" We prove in this section that given a map $f : X \to \mathbb{K}^n$ with generically finite fibers, there is a filtration on $\mathbb{K}[X]$ such that the corresponding completion preserves map f at ∞. When pondering about a completion ψ preserving fibers of a map f, it is natural to try to understand how large in $f(X)$ is the set S_ψ defined as the set of all points a in $f(X)$ such that ψ preserves $f^{-1}(a)$ at ∞. The structure of S_ψ, in general, is a mystery to us. We show by means of examples that S_ψ can be a proper Zariski closed or a proper Zariski open subset of $f(X)$. In the special class of a semidegree, we show later in section 3.1 that $S_\psi = f(X)$ if S_ψ is not empty.

We introduce degree like functions corresponding to filtrations in section 2.1 and present examples of completions determined by semidegrees and subdegrees. In particular we show that normal projective toric varieties come from subdegrees on the ring of Laurent polynomials. In section 2.2 we prove our main results on projective completions determined by subdegrees. Our first theorem classifies the filtrations determined by semi- and subdegrees. As a corollary we deduce that for a completion $\psi : X \hookrightarrow \bar{X}$ given by a subdegree δ, the irreducible components of $X_\infty := \bar{X} \setminus X$ are in a one-to-one correspondence with the unique minimal collection of semidegrees defining δ. We also show that $\bar{X}$ is *nonsingular in codimension one at infinity*, i.e. for each irreducible component V of X_∞ the local ring $\mathcal{O}_{V,\bar{X}}$ of $\bar{X}$ along V is regular and hence a discrete valuation ring; moreover, the semidegree associated to V is proportional to the valuation associated to $\mathcal{O}_{V,\bar{X}}$. Given an arbitrary completion which comes from a filtration and preserves the intersection at ∞ of a collection of subvarieties, we show in our (main) existence theorem that there is a completion determined by a subdegree which preserves the intersection at ∞ of the subvarieties in the collection. We make use of the latter to conclude the existence of completions determined by subdegrees which preserve a given generically finite polynomial map at ∞. Then we show that 'subdegree' can not be replaced by a 'semidegree' in the preceding result: we provide an example of a quasifinite map $f : \mathbb{K}^2 \to \mathbb{K}^2$ which does not admit any semidegree on $\mathbb{K}[x_1, x_2]$ such that the corresponding completion of $\mathbb{K}^2$ preserves $\{f_1, f_2\}$ at ∞ over any point in $\mathbb{K}^2$.

In section 3.1 we prove the Bezout theorem for a semidegree δ in terms of the degree D

of the resulting projective variety in an appropriate ambient space. We give a description of D in terms of the volume of a convex body in $\mathbb{R}^n$ using [17]. We describe in section 3.2 an explicit construction of *iterated* semidegrees. We then present a very simple formula for D in the case that $X = \mathbb{K}^n$ and δ is an iterated semidegree on $\mathbb{K}[x_1, \ldots, x_n]$ constructed by means of finitely many iterations starting with the case of a weighted degree as the initial semidegree. In section 3.3 we give an (implicit) Bezout-type bound for a wide class of subdegrees: if $f : X \to \mathbb{K}^n$ is a polynomial with generically finite fibers and δ is a subdegree such that each of its associated semidegrees takes *positive* value on each of the components of f, then we derive a formula for the size of generic fibers of f in terms of the degree of a completion of X constructed from f and δ. (We expect the latter degree to be the mixed volume of n convex sets for which we have an explicit construction; but this is a work in progress.)

In the appendix, together with a precise statement of Bernstein's theorem, we prove that the non-degeneracy condition in his theorem is equivalent to the condition that the completion $(\mathbb{C}^*)^n \hookrightarrow X_{\mathcal{P}}$ preserves $\{f_1, \ldots, f_n\}$ at ∞ over 0, where $X_{\mathcal{P}}$ is the toric variety associated with the *Minkowski sum* $\mathcal{P}$ of the Newton polytopes of components of f.

Chapter 1

Intersection preserving completions determined by filtrations

We start this chapter by describing how filtrations on the coordinate ring of a given affine variety X determine projective completions of X. 'Preserving intersections' in the title stands for the intersection of completions being the same as the completion of the intersection for a family of subvarieties of X. We show that there exist projective completions of X (determined by filtrations) which preserve the intersection of a given finite family of subvarieties of X with a finite intersection. For a polynomial map $P := (P_1, \ldots, P_n) : X \to \mathbb{K}^n$ with generically finite fibers, we prove that there are completions of the source which preserve the intersection of the component hypersurfaces $\{P_j = a_j\}$ for generic $(a_1, \ldots, a_n) \in \mathbb{K}^n$. Moreover, we show that there are completions of X which preserve the intersection of component hypersurfaces of $\xi \circ P$ for *any* non-degenerate linear transformation ξ of $\mathbb{K}^n$.

1.0 Background

In this section we recall definitions and theorems from basic algebraic geometry that we use throughout the text.

1.0.1 Affine Varieties

Let $\mathbb{K}$ be an algebraically closed field. The *affine n-space* over $\mathbb{K}$ is the set $\mathbb{K}^n$. Let $x_1, \ldots, x_n$ be the coordinate functions on $\mathbb{K}^n$. A *Zariski closed* subset of $\mathbb{K}^n$ is a set of solutions of a collection of polynomials in $(x_1, \ldots, x_n)$. The *Zariski topology* on $\mathbb{K}^n$ is the topology for which the closed sets are precisely the Zariski closed subsets of $\mathbb{K}^n$. A Zariski closed subset X of $\mathbb{K}^n$ is an *affine algebraic variety* or, in short an *affine variety* if it is also *irreducible*, i.e. it is not the union of two proper Zariski closed subsets. Let $\phi : X \to Y \subseteq \mathbb{K}^m$ be a map between affine varieties. ϕ is called a *morphism* or a *regular map* if it is the restriction to X of a polynomial map from $\mathbb{K}^n$ to $\mathbb{K}^m$. The *coordinate ring* $\mathbb{K}[X]$ of X is the ring of the $\mathbb{K}$-valued functions on X which are restrictions of polynomials in $(x_1, \ldots, x_n)$. Equivalently, $\mathbb{K}[X]$ is the quotient of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ by the ideal of polynomials vanishing on all points of X. The starting point of modern algebraic geometry is Hilbert's *Nullstellensatz*, which gives a one-to-one correspondence between the Zariski closed subsets of an affine algebraic variety and radical ideals of its coordinate ring in the following way:

$$\begin{array}{ccc} \textit{Zariski closed subsets of a variety } X & \longleftrightarrow & \textit{Radical ideals of } \mathbb{K}[X] \\ V & \longrightarrow & I(V) := \{f : f(x) = 0 \text{ for all } x \in V\} \\ V(I) := \{x : f(x) = 0 \text{ for all } f \in I\} & \longleftarrow & I \end{array}$$

In particular, the points of X correspond to the maximal ideals of $\mathbb{K}[X]$, and subvarieties of X correspond to prime ideals of $\mathbb{K}[X]$. A corollary of this is the following useful result:

Theorem 1.0.1 ([14, Theorem I.3.5])**.** *The correspondence $X \to \mathbb{K}[X]$ gives an equivalence of the category of affine algebraic varieties over $\mathbb{K}$ and regular maps with the category of finitely generated domains over $\mathbb{K}$ and $\mathbb{K}$-algebra homomorphisms.*

In particular, X and Y are isomorphic as affine varieties if and only if $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ are isomorphic as $\mathbb{K}$-algebras, and choosing different sets of generators of $\mathbb{K}[X]$ gives different embeddings of X into possibly different affine spaces.

Let A be a finitely generated domain over $\mathbb{K}$. Let $\operatorname{Spec} A$ be the space $\{\mathfrak{p} : \mathfrak{p}$ is a prime ideal of $A\}$ together with the Zariski topology, in which closed sets are of the form $V(\mathfrak{a}) := \{\mathfrak{p} : \mathfrak{p} \supseteq a\}$ for some ideal $\mathfrak{a}$ of A. As f varies in A, the complement of $V(f)$, which we denote by $D(f)$, form a basis for the Zariski topology on $\operatorname{Spec} A$. For each $f \in A$, let $A_f := \{a/f^k : a \in A, k \in \mathbb{N}\}$ be the *localization* of A with respect to f. Note that for each $f, g \in A$, there is a natural map $A_f \to A_{fg}$ which maps $a/f^k \mapsto ag^k/(fg)^k$. Since $D(f) \cap D(g) = D(fg)$ for all $f, g \in A$, it follows that there is a unique sheaf $\mathcal{O}_{\operatorname{Spec} A}$, called the *structure sheaf*, on $\operatorname{Spec} A$ such that its ring of sections over each $D(f)$ is A_f. Moreover, it follows from theorem 1.0.1 that up to an isomorphism there is a unique affine variety X, such that X together with the sheaf $\mathcal{O}_X$ of regular functions on X completely determines $\operatorname{Spec} A$ and $\mathcal{O}_{\operatorname{Spec} A}$. By an abuse of notation we will write $\operatorname{Spec} A$ also for this X, whenever the meaning will be clear from the context.

1.0.2 Projective Varieties

The *projective n-space* over $\mathbb{K}$, denoted $\mathbb{P}^n(\mathbb{K})$, or simply $\mathbb{P}^n$, is the quotient of $\mathbb{K}^{n+1} \setminus \{0\}$ under the equivalence relation given by $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for all $\lambda \in \mathbb{K} \setminus \{0\}$. Denote the equivalence class of $(a_0, \ldots, a_n)$ by the *homogeneous coordinates* $[a_0 : \cdots : a_n]$. Let $U_i := \{[a_0 : \cdots : a_n] : a_i \neq 0\}$. For each i, identify U_i with $\mathbb{K}^n$ via the mapping

$$[a_0 : \cdots : a_n] \mapsto (\frac{a_0}{a_i}, \ldots, \frac{\widehat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i}).$$

The Zariski topology on $\mathbb{P}^n$ is defined by declaring a subset X to be closed if and only if $X \cap U_i$ is Zariski closed in U_i for each i. As in the affine case, a Zariski closed subset X of $\mathbb{P}^n$ is a *projective variety* if X is also irreducible.

Let $x_0, \ldots, x_n$ be the coordinates on $\mathbb{K}^{n+1}$. Let $f = \sum a_\alpha x^\alpha$ be an element in $\mathbb{K}[x_0, \ldots, x_n]$, where $\alpha = (\alpha_0, \ldots, \alpha_n)$ varies over $(\mathbb{Z}_+)^{n+1}$, and we use the multi-index notation that $x^\alpha := x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. Recall that f is called *homogeneous of degree d* if

$\sum_{i=0}^{n} \alpha_i = d$ for each α such that $a_\alpha \neq 0$, and an ideal of $\mathbb{K}[x_0, \ldots, x_n]$ is homogeneous if it has a set of homogeneous generators. It turns out that a subset X of $\mathbb{P}^n$ is Zariski closed if and only if there are homogeneous polynomials $f_1, \ldots, f_k \in \mathbb{K}[x_0, \ldots, x_n]$ such that $X = \{[a_0 : \cdots : a_n] : f_i(a_0, \ldots, a_n) = 0 \text{ for all } i\}$. Let $I(X)$ be the homogeneous ideal of $\mathbb{K}[x_0, \ldots, x_n]$ generated by the homogeneous polynomials which vanish at all points of X. Then X is irreducible, i.e. a projective variety, if and only if $I(X)$ is prime. The quotient $S(X) := \mathbb{K}[x_0, \ldots, x_n]/I(X)$ is called the *homogeneous coordinate ring* of X. The projective version of Nullstellensatz gives a one-to-one correspondence, defined exactly as in 1.0.1, between subvarieties of X and radical homogeneous ideals *properly contained* in the *irrelevant ideal* $S(X)_+$ of $S(X)$, where $S(X)_+$ is the image in $S(X)$ of the ideal of $\mathbb{K}[x_0, \ldots, x_n]$ generated by $x_0, \ldots, x_n$.

A *quasi-projective* variety is a Zariski open subset of a projective variety. A quasi-projective variety X is called *complete* if for every quasi-projective variety Y the projection map $p : X \times Y \to Y$ is closed, i.e. p maps Zariski closed sets onto Zariski closed sets. Complete spaces are algebraic analogue of compact spaces - if X is a 'reasonable' topological space, say completely regular or with a countable basis of open sets, one can show that X is complete in the above sense if and only if X is compact. The *fundamental theorem of elimination theory* says that $\mathbb{P}^n$ is complete for all n. It follows that any projective variety, being a Zariski closed subset of some $\mathbb{P}^n$, is also complete. A *projective completion* of an affine variety Y is an *open immersion* $\psi : Y \hookrightarrow X$ of Y into a projective variety X, where 'ψ is an open immersion' means that $\psi(Y)$ is a Zariski open subset of X and ψ is an isomorphism of algebraic varieties between Y and $\psi(Y)$. Whenever the map ψ is clear from the context, we will simply say that X is a projective completion of Y.

1.0.3 Graded Rings

A *graded ring* is a ring S, together with a decomposition $S = \bigoplus_{d\in\mathbb{Z}} S_d$ into a direct sum of abelian groups S_d such that for any $d, e \in \mathbb{Z}$, $S_d \cdot S_e \subseteq S_{d+e}$. The grading is *non-negative* if $S_d = 0$ for all $d < 0$. An element of S_d is called a *homogeneous element of degree* d, and an ideal $\mathfrak{a}$ of S is a *homogeneous ideal* if $\mathfrak{a} = \bigoplus_{d\in\mathbb{Z}}(\mathfrak{a} \cap S_d)$. The *irrelevant ideal* of S is the homogeneous ideal generated by $S_+ := \bigoplus_{d\geq 1} S_d$.

As in subsection 1.0.1, we now associate to every graded ring S a topological space, together with a sheaf of rings. Mimicing the situation of a projective variety in subsection 1.0.2, define $\operatorname{Proj} S$ to be the set of all homogeneous prime ideals of S which do not contain all of S_+. Endow $\operatorname{Proj} S$ with the Zariski topology, where closed sets are of the form $V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \supseteq \mathfrak{a}\}$ for homogeneous ideals $\mathfrak{a}$ of S. As f varies over the homogeneous elements of *positive* degree in S, the sets $D(f) := \{\mathfrak{p} \in \operatorname{Proj} S : f \notin \mathfrak{p}\}$ form a basis of the Zariski topology on $\operatorname{Proj} S$. Let f be a homogeneous element of degree $d > 0$ in S and let $S_{(f)} := \{s/f^k : s \in S_{kd}, k \in \mathbb{Z}\}$ be the subring of elements of degree 0 in the localized ring S_f. As in the affine case, there is a unique sheaf $\mathcal{O}_{\operatorname{Proj} S}$ on $\operatorname{Proj} S$ such that $S_{(f)}$ is the ring of sections over $D(f)$.

Theorem 1.0.2 ([14, Theorems II.2.5])**.** *Let S be a graded ring which is also a finitely generated domain over $\mathbb{K}$ (in particular, each S_d is a $\mathbb{K}$-vector space). Assume the grading of S is non-negative.*

1. *Let f be a homogeneous element of positive degree in S. Then there is an isomorphism* $\operatorname{Spec} S_{(f)} \cong (D(f), \mathcal{O}_{\operatorname{Proj} S}|_{D(f)})$*, where the isomorphism is that of* ringed spaces*, which means that there is a pair $(\phi, \phi^\sharp)$ of a homeomorphism $\phi : D(f) \to$* $\operatorname{Spec} S_{(f)}$ *and an isomorphism of sheaves $\phi^\sharp : \mathcal{O}_{\operatorname{Spec} S_{(f)}} \to \phi_*(\mathcal{O}_{\operatorname{Proj} S}|_{D(f)})$.*
2. *Up to an isomorphism there is a unique quasi-projective variety X such that X together with the sheaf $\mathcal{O}_X$ of regular functions on X completely determines* $\operatorname{Proj} S$ *and $\mathcal{O}_{\operatorname{Proj} S}$. By an abuse of notation, this X will also be denoted by* $\operatorname{Proj} S$.
3. *If in addition $S_0 = \mathbb{K}$, then* $\operatorname{Proj} S$ *is a projective variety.*

Example 1.0.3. Let $d_0, \ldots, d_n$ be positive integers and let R be the polynomial ring $\mathbb{K}[x_0, \ldots, x_n]$ endowed with the grading given by the *weighted degree* which associates weight d_i to x_i. In other words, $R = \bigoplus_{d \geq 0} R_d$ with R_d being the $\mathbb{K}$-vector space generated by monomials $x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ such that $\sum \alpha_j d_j = d$. Then $\operatorname{Proj} R$ is the n-dimensional *weighted projective space* $\mathbb{P}^n(\mathbb{K}; d_0, \ldots, d_n)$. As a set it can be identified with the quotient of $\mathbb{K}^{n+1} \setminus \{0\}$ under the equivalence relation given by $(a_0, \ldots, a_n) \sim (\lambda^{d_0} a_0, \ldots, \lambda^{d_n} a_n)$ for all $\lambda \in \mathbb{K} \setminus \{0\}$. In particular, for $d_0 = \cdots = d_n = 1$, $\operatorname{Proj} R$ is the usual projective space $\mathbb{P}^n(\mathbb{K})$.

Example 1.0.4. Let $d_0 = 1$ and $R, d_1, \ldots, d_n$ be as in example 1.0.3. Then $R_{(x_0)} \cong \mathbb{K}[y_1, \ldots, y_n]$ via the mapping $y_i \mapsto x_i/(x_0)^{d_i}$. It follows via theorems 1.0.1 and 1.0.2 that the Zariski open subset D_{x_0} of $\operatorname{Proj} R$ is isomorphic to $\mathbb{K}^n$ as algebraic varieties - in other words $\operatorname{Proj} R$ is a *projective completion* of $\mathbb{K}^n$.

Example 1.0.5. Let S be as in assertion 3 of theorem 1.0.2 (in particular, $S_0 = \mathbb{K}$). Let $f_0, \ldots, f_k$ be a set of generators of S as a $\mathbb{K}$-algebra, with $\deg(f_i) = d_i > 0$. Let R be as in example 1.0.3. Then there is a surjective homomorphism $\phi : R \to S$ of graded rings which maps $x_i \mapsto f_i$, and therefore $S \cong R/I$, where $I := \ker \phi$. Let X be the subvariety of the weighted projective space $\mathbb{P}^n(\mathbb{K}; d_0, \ldots, d_k)$ corresponding to the homogeneous ideal I. It follows that the *homogeneous coordinate ring* of X is isomorphic to S and hence $\operatorname{Proj} S \cong X$.

Example 1.0.6. Let S, f_i and d_i, $0 \leq i \leq k$, be as in example 1.0.5 and $d > 0$. Let the d-th truncated ring $S^{[d]} \subseteq S$ be defined by $S^{[d]} := \bigoplus_{d|k} S_k = \bigoplus_{k \geq 0} S_{kd}$. Then $\mathfrak{p} \mapsto \mathfrak{p} \cap S^{[d]}$ gives a homeomorphism between underlying sets of $\operatorname{Proj} S$ and $\operatorname{Proj} S^{[d]}$, and moreover, for each homogeneous $f \in S_+$, $S_{(f)} \cong (S^{[d]})_{(f^d)}$. It follows that $\operatorname{Proj} S \cong \operatorname{Proj} S^{[d]}$. As in the preceding example, choosing a set of generators of $S^{[d]}$, one can embed $\operatorname{Proj} S$ as a subvariety of an appropriate weighted projective space. Such an embedding is called a *d-uple embedding.* Moreover, there exists d such that $S^{[d]}$ is generated as a $\mathbb{K}$-algebra by

$(S^{[d]})_1 = S_d$ (e.g. it suffices to take $d = (k+1)d'$, where d' is the least common multiple of $d_0, \ldots, d_k$ [27, Lemma in section III.8]). In that case the d-uple embedding embeds $\operatorname{Proj} S$ into a usual projective space.

1.1 Filtrations

Throughout this section A will be a finitely generated algebra over an algebraically closed field $\mathbb{K}$.

Definition. A filtration $\mathcal{F}$ on A is a family $\{F_i : \ i \in \mathbb{Z}\}$ of $\mathbb{K}$-vector subspaces of A such that

1. $F_i \subseteq F_{i+1}$ for all $i \in \mathbb{Z}$
2. $1 \in F_0 \setminus F_{-1}$
3. $A = \bigcup_{i \in \mathbb{Z}} F_i$ and
4. $F_i F_j \subseteq F_{i+j}$ for all $i, j \in \mathbb{Z}$.

Remark. We introduce condition $1 \notin F_{-1}$ in order to exclude the trivial filtration, i.e. filtration $\{F_i := A\}_{i \in \mathbb{Z}}$.

Filtration $\mathcal{F}$ is called *non-negative* if $F_i = 0$ for all $i < 0$. Associated to each filtration $\mathcal{F}$ there are two graded rings:

$$A^{\mathcal{F}} := \bigoplus_{i \in \mathbb{Z}} F_i \quad \text{and}$$
$$\operatorname{gr} A^{\mathcal{F}} := \bigoplus_{i \in \mathbb{Z}} (F_i / F_{i-1}).$$

We denote a copy of $f \in F_d$ in the d-th graded component of $A^{\mathcal{F}}$ by $(f)_d$. $A^{\mathcal{F}}$ is given the structure of a graded $\mathbb{K}$-algebra with multiplication defined by:

$$(\sum_d (f_d)_d)(\sum_e (g_e)_e) := \sum_k \sum_{d+e=k} (f_d g_e)_k.$$

$\mathcal{F}$ is called a *finitely generated* filtration if $A^{\mathcal{F}}$ is a finitely generated $\mathbb{K}$-algebra.

Let t be an indeterminate over A. Then there is an isomorphism

$$A^{\mathcal{F}} \cong \sum_{i \in \mathbb{Z}} F_i t^i \subseteq A[t, t^{-1}] \tag{1.1}$$

which maps $(1)_1 \mapsto t$. The following property of $A^{\mathcal{F}}$ is a straightforward corollary of this isomorphism.

Lemma 1.1.1. *$A^{\mathcal{F}}$ is a domain if and only if A is a domain.* □

It follows from theorem 1.0.2 that $\operatorname{Proj} A^{\mathcal{F}}$ is a projective, and hence complete, variety if $\mathcal{F}$ is non-negative, finitely generated and $F_0 = \mathbb{K}$. In view of this fact we make the following

Definition. A filtration $\mathcal{F} = \{F_d\}_{d \in \mathbb{Z}}$ on A is called *complete* if it is non-negative, finitely generated and $F_0 = \mathbb{K}$.

The following proposition connects filtrations on A with projective completions of $\operatorname{Spec} A$. Recall from section 1.0.3 that the d-th truncated subring of a graded ring $S = \bigoplus_{k \in \mathbb{Z}} S_k$ is $S^{[d]} := \bigoplus_{k \in \mathbb{Z}} S_{kd}$.

Proposition 1.1.2. *If A is a $\mathbb{K}$-algebra and $\mathcal{F}$ is a non-negative filtration on A, then there is an open immersion $\psi_{\mathcal{F}}$ of $\operatorname{Spec} A$ onto a dense open subvariety of $\operatorname{Proj} A^{\mathcal{F}}$. The complement of $\operatorname{Spec} A$ in $\operatorname{Proj} A^{\mathcal{F}}$ is a hypersurface. Conversely, given any non-negatively graded $\mathbb{K}$-algbera $S = \bigoplus_{i \geq 0} S_i$ and an open immersion $\phi : \operatorname{Spec} A \hookrightarrow \operatorname{Proj} S$ such that $\operatorname{Spec} A$ is dense in $\operatorname{Proj} S$ and $\operatorname{Proj} S \setminus \operatorname{Spec} A$ is a hypersurface, there is a non-negative filtration $\mathcal{F}$ on A and a commutative diagram as follows:*

$$\begin{array}{ccc} \operatorname{Spec} A & \xrightarrow{\cong} & \operatorname{Spec} A \\ \downarrow{\scriptstyle \psi_{\mathcal{F}}} & & \downarrow{\scriptstyle \phi} \\ \operatorname{Proj} A^{\mathcal{F}} & \longrightarrow & \operatorname{Proj} S \end{array}$$

where the top horizontal map is the identity and the bottom horizontal map is a closed immersion induced by an isomorphism of $A^{\mathcal{F}}$ with $S^{[d]}/I$ for some $d > 0$ and some ideal I of $S^{[d]}$ contained in the nilradical of $S^{[d]}$.

Remark.

- By a *hypersurface* in $\operatorname{Proj} S$ for an arbitrary graded ring S we just mean a closed subvariety given by $V(f)$ for some *homogeneous* $f \in S$.
- The first part of the proposition is well known.

Proof. Recall from section 1.0.3 that basic open sets in $\operatorname{Proj} A^{\mathcal{F}}$ are given by $D(G) = \{Q \in \operatorname{Proj} A^{\mathcal{F}} | G \notin Q\} = \operatorname{Spec}(A^{\mathcal{F}})_{(G)}$, where G ranges over the homogeneous elements in $A^{\mathcal{F}}$, and $(A^{\mathcal{F}})_{(G)}$ is the subring of the local ring $(A^{\mathcal{F}})_G$ consisting of degree zero homogeneous elements. Identify $A^{\mathcal{F}}$ with $\sum_{i\in\mathbb{Z}} F_i t^i$ via (1.1). It follows that $(A^{\mathcal{F}})_{(t)} = \{gt^d/t^d : g \in F_d \subseteq A,\ d \geq 0\} \cong \{g : g \in F_d \subseteq A\} = A$ as a subring of $A[t, t^{-1}]$; hence $\operatorname{Spec} A \cong \operatorname{Spec}(A^{\mathcal{F}})_{(t)} \cong D(t)$ and $\operatorname{Proj} A^{\mathcal{F}} \setminus \operatorname{Spec} A = V(t)$. But by lemma 1.1.1 $A^{\mathcal{F}}$ is a domain and it is a general fact that for any graded ring S and any non zero-divisor $G \in S$, $\operatorname{Proj} S \setminus V(G)$ is dense in $\operatorname{Proj} S$. Therefore $\operatorname{Spec} A$ is dense in $\operatorname{Proj} A^{\mathcal{F}}$ and we proved the first assertion of the proposition.

Let $S = \bigoplus_{i\geq 0} S_i$ be a graded $\mathbb{K}$-algebra and $\phi : \operatorname{Spec} A \to \operatorname{Proj} S$ be an open immersion of $\operatorname{Spec} A$ onto a dense open subvariety of $\operatorname{Proj} S$ of the form $D(f)$ for some homogeneous $f \in S$. Let the degree of f be d. Recall from example 1.0.5 that $\operatorname{Proj} S \cong \operatorname{Proj} S^{[d]}$. Hence replacing S with $S^{[d]}$, we may assume that degree of f is 1. By theorem 1.0.1 the isomorphism of varieties $\phi : D(f) \cong \operatorname{Spec} S_{(f)}$ induces an isomorphism of rings $\phi^* : S_{(f)} \cong A$. Now, for each $g \in A$, $(\phi^*)^{-1}(g) = a/f^k$ for some $k \geq 0$ and $a \in S_k$. Define

$$F_k := \phi^*(S_k/f^k) = \{g \in R|\ (\phi^*)^{-1}(g) \in S_k/f^k\}.$$

Then it is easy to see that $\mathcal{F} = \{F_i\}_{i\geq 0}$ is a filtration on A. By means of this filtration we construct, as usual, the ring $A^{\mathcal{F}} := \bigoplus_{d\geq 0} F_d$. Map $\phi^* : S_{(f)} \to A$ induces a map $\Phi^* : S \to A^{\mathcal{F}}$ which sends each $g \in S_d$ to $(\phi^*(g/f^d))_d$, i.e. to the copy of $\phi^*(g/f^d)$ in $F_d \subseteq A^{\mathcal{F}}$. Consequently Φ^* is a surjective homomomorphism of graded rings, so that the induced map $\Phi : \operatorname{Proj} A^{\mathcal{F}} \to \operatorname{Proj} S$ is a closed immersion [14, Exercise II.3.12(a)] and $\operatorname{Proj} A^{\mathcal{F}} \cong \operatorname{Proj} \bar{S}$, where $\bar{S} := S/\ker \Phi^*$.

It remains to show that with $I := \ker \Phi^*$, ideal $I \subseteq \eta(S)$, where $\eta(S)$ is the nilradical of S. Indeed, let $g \in S_d$ such that $\Phi^*(g) = (\phi^*(g/f^d))_d = 0 \in A^{\mathcal{F}}$. Then $\phi^*(g/f^d) = 0 \in A$. Since $\phi^* : S_{(f)} \to A$ is an isomorphism, it follows that $g/f^d = 0 \in S_{(f)}$ so that there is some $k \geq 0$ such that $f^k g = 0 \in S$. But then every prime ideal in S has to contain f or g, i.e. $D(f) \cap D(g) = \emptyset$. Since $D(f)$ ($\cong \operatorname{Spec} A$) is dense in $\operatorname{Proj} S$, it follows that $D(g) = \emptyset$, so that $g \in \eta(S)$.

To see that the isomorphism $\operatorname{Proj} A^{\mathcal{F}} \cong \operatorname{Proj} \bar{S}$ is the identity map when restricted to $\operatorname{Spec} A$, note that maps $\phi : \operatorname{Spec} A \to \operatorname{Proj} S$ and $\psi_{\mathcal{F}} : \operatorname{Spec} A \to \operatorname{Proj} A^{\mathcal{F}}$ are completely determined by the corresponding pull back maps $\phi^* : S_{(f)} \to A$ and $\psi^*_{\mathcal{F}} : (A^{\mathcal{F}})_{(t)} \to A$ on the localizations of the respective graded $\mathbb{K}$-algebras, where as before $t := (1)_1 \in A^{\mathcal{F}}$. The latter two maps give rise to the following commutative diagrams:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \uparrow{\scriptstyle \phi^*} & & \uparrow{\scriptstyle \psi^*_{\mathcal{F}}} \\ S_{(f)} & \xrightarrow{\Phi^*} & (A^{\mathcal{F}})_{(t)} \end{array} \qquad \begin{array}{ccc} \phi^*(g/f^d) & \xmapsto{1_A} & \phi^*(g/f^d) \\ \uparrow{\scriptstyle \phi^*} & & \uparrow{\scriptstyle \psi^*_{\mathcal{F}}} \\ g/f^d & \xmapsto{\Phi^*} & \frac{(\phi^*(g/f^d))_d}{t^d} \end{array}$$

Since the top horizontal map and both of the vertical maps are isomorphisms, it follows that the bottom horizontal map is an isomorphism as well, which concludes the proof of proposition 1.1.2. $\square$

Before we proceed further, let us examine the structure of the complement of $\operatorname{Spec} A$ in $\operatorname{Proj} A^{\mathcal{F}}$ more closely. From the proof of proposition 1.1.2, we see that $\operatorname{Proj} A^{\mathcal{F}} \setminus \operatorname{Spec} A$ is the closed subspace $V(t) := \{P \in \operatorname{Proj} A^{\mathcal{F}} | \ t \in P\}$. Consider the natural projection $\pi : A^{\mathcal{F}} \to \operatorname{gr} A^{\mathcal{F}}$. Map π is a surjective homomorphism of graded rings and therefore induces a closed immersion $\pi^* : \operatorname{Proj}(\operatorname{gr} A^{\mathcal{F}}) \to \operatorname{Proj} A^{\mathcal{F}}$, given by $\mathfrak{p} \mapsto \pi^{-1}(\mathfrak{p})$.

Claim. $\ker \pi = \langle t \rangle$, *where* $\langle t \rangle$ *is the ideal generated by* t *in* $A^{\mathcal{F}}$.

Proof. Since $t := (1)_1$ and $1 \in F_0$, it follows that $t \in \ker \pi$, so that $\ker \pi \supseteq \langle t \rangle$. Also, π is a homomorphism of graded ring. Hence $\ker \pi$ is a homogeneous ideal, i.e. $\ker \pi$ is generated by homogeneous elements from $A^{\mathcal{F}}$. Pick a homogeneous element G in the d-th graded component of $\ker \pi$. Then $G = (g)_d$ for some $g \in F_{d-1}$. But then $G = (g)_d = (g)_{d-1}(1)_1 = (g)_{d-1}t$, so that $G \in \langle t \rangle$. Therefore, $\ker \pi \subseteq \langle t \rangle$. □

It follows that $\pi^*(\mathrm{Proj}(\mathrm{gr}\, A^{\mathcal{F}})) = V(t)$ [14, Exercise II.3.12(b)]. Thus we have proven the following

Proposition 1.1.3. $\mathrm{Proj}\, A^{\mathcal{F}} \setminus \mathrm{Spec}\, A = \mathrm{Proj}(\mathrm{gr}\, A^{\mathcal{F}})$. □

We combine the results of 1.1.1, 1.1.2 and 1.1.3 in

Theorem 1.1.4 (see [25, Theorem 1.1] and [26, Theorem 1.1.4])**.** *Let A be a $\mathbb{K}$-algebra which is a domain and $\mathcal{F}$ be a filtration on A. If $\mathcal{F}$ is complete, then* $\mathrm{Proj}\, A^{\mathcal{F}}$ *is a projective variety over $\mathbb{K}$ and the map*

$$\psi_{\mathcal{F}} : p \to \bigoplus_{d \geq 0} p \cap F_d$$

gives an open immmersion of $\mathrm{Spec}\, A$ *onto a dense open subscheme of* $\mathrm{Proj}\, A^{\mathcal{F}}$*. The complement of* $\mathrm{Spec}\, A$ *in* $\mathrm{Proj}\, A^{\mathcal{F}}$ *is the hypersurface $V((1)_1)$, which also is the image of the closed immersion* $\pi^* : \mathrm{Proj}(\mathrm{gr}\, A^{\mathcal{F}}) \hookrightarrow \mathrm{Proj}(A^{\mathcal{F}})$ *induced by the natural homomorphism* $\pi : A^{\mathcal{F}} \to \mathrm{gr}\, A^{\mathcal{F}}$*. Conversely, given any graded ring S and an open immersion* $\phi : \mathrm{Spec}\, A \hookrightarrow \mathrm{Proj}\, S$ *such that* $\mathrm{Spec}\, A$ *is dense in* $\mathrm{Proj}\, S$ *and* $\mathrm{Proj}\, S \setminus \mathrm{Spec}\, A$ *is a hypersurface, there is a filtration $\mathcal{F}$ on A and a commutative diagram as follows:*

$$\begin{array}{ccc} \mathrm{Spec}\, A & \xrightarrow{\cong} & \mathrm{Spec}\, A \\ \downarrow{\scriptstyle \psi_{\mathcal{F}}} & & \downarrow{\scriptstyle \phi} \\ \mathrm{Proj}\, A^{\mathcal{F}} & \longrightarrow & \mathrm{Proj}\, S \end{array}$$

where the top horizontal map is the identity and the bottom horizontal map is a closed immersion induced by an isomorphism of graded rings $A^{\mathcal{F}} \cong S^{[d]}/\eta(S^{[d]}))$ for some $d > 0$, where $\eta(S^{[d]})$ is the nilradical of $S^{[d]}$. □

Recall from section 1.0.3 the concept of the *homogeneous coordinate ring* of a closed subvariety of a n-dimensional weighted projective space. Choosing a set of generators of $A^{\mathcal{F}}$ gives an embedding of $\operatorname{Proj} A^{\mathcal{F}}$ into a weighted projective space such that the homogeneous coordinate ring of the image is $A^{\mathcal{F}}$. The following corollary is a rewording of theorem 1.1.4 in terms of these coordinates.

Corollary 1.1.5. *Let X be an affine variety over $\mathbb{K}$ and $\mathcal{F}$ be a complete filtration on $\mathbb{K}[X]$. Pick $f_1, \ldots, f_N \in \mathbb{K}[X]$ such that $(1)_1, (f_1)_{d_1}, \ldots, (f_N)_{d_N}$ generate $\mathbb{K}[X]^{\mathcal{F}}$ as a $\mathbb{K}$-algebra. Then the map $\phi : X \to \mathbb{K}^N$ given by $\phi(x) := (f_1(x), \ldots, f_N(x))$ is a closed immersion, and there exists a commutative diagram as follows:*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \mathbb{K}^N \\ {\scriptstyle \psi_{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \iota} \\ \operatorname{Proj} \mathbb{K}[X]^{\mathcal{F}} & \xrightarrow{\Phi} & \mathbb{P}^N(\mathbb{K}; 1, d_1, \ldots, d_N) \end{array}$$

where ι is the natural injection of $\mathbb{K}^N$ into the weighted projective space $\mathbb{P}^N(\mathbb{K}; 1, d_1, \ldots, d_N)$, and Φ is a closed immersion that maps $\operatorname{Proj} \mathbb{K}[X]^{\mathcal{F}}$ isomorphically onto the closure $\bar{X}$ of X in $\mathbb{P}^N(\mathbb{K}; 1, d_1, \ldots, d_N)$. In fact, morphism Φ is induced by an isomorphism between $\mathbb{K}[X]^{\mathcal{F}}$ and the homogeneous coordinate ring of $\bar{X}$. Conversely, given any closed immersion $\phi : X \hookrightarrow \mathbb{K}^N$, and any N positive integers $d_1, \ldots, d_N$, there exists a complete filtration $\mathcal{F}$ on $\mathbb{K}[X]$ and a closed immersion $\Phi : \operatorname{Proj} \mathbb{K}[X]^{\mathcal{F}} \hookrightarrow \mathbb{P}^N(\mathbb{K}; 1, d_1, \ldots, d_N)$ such that the above diagram is commutative. Moreover, morphism Φ is induced by a graded ring isomorphism between $\mathbb{K}[X]^{\mathcal{F}}$ and the homogeneous coordinate ring of $\bar{X}$ in $\mathbb{P}^N(\mathbb{K}; 1, d_1, \ldots, d_N)$. □

1.2 Examples

In this section we work out some examples of complete filtrations on the coordinate rings of affine varieties, and determine the corresponding completions. Throughout this section X will denote an affine variety, A will denote its coordinate ring, $\mathcal{F} = \{F_d : d \geq 0\}$ will be a filtration on A and $X^{\mathcal{F}} := \operatorname{Proj} A^{\mathcal{F}}$ will denote the corresponding completion of X. For the first four examples we set $X = \mathbb{K}^n$ and $A = \mathbb{K}[x_1, \ldots, x_n]$.

Example 1.2.1. Define $\mathcal{F}$ on A by letting F_d be the set of polynomials of degree less than or equal to d. Identification of F_d with the set of *homogeneous* polynomials in $x_0, \ldots, x_n$ of degree d gives an graded ring isomorphism $A^{\mathcal{F}} \cong \mathbb{K}[x_0, \ldots, x_n]$, where the latter is graded by the usual degree. Hence $X^{\mathcal{F}}$ is the usual projective space $\mathbb{P}^n(\mathbb{K})$.

Example 1.2.2. Let $d_1, \ldots, d_n$ be any n positive integers. Let F_d be the $\mathbb{K}$-linear span of all the monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ such that $\sum \alpha_i d_i \leq d$. Set $d_0 := 1$. Then F_d can be identified with the set of *weighted homogeneous* polynomials in $x_0, \ldots, x_n$ of weighted degree d, where weight of x_i is d_i for each i. Thus $A^{\mathcal{F}}$ is again isomorphic to $\mathbb{K}[x_0, \ldots, x_n]$, but the grading in this case is induced by the weighted degree $(d_0, d_1, \ldots, d_n)$, and $X^{\mathcal{F}}$ is the weighted projective space $\mathbb{P}^n(\mathbb{K}; d_0, d_1, \ldots, d_n)$ which we have already seen in example 1.0.4.

Example 1.2.3. Let F_k be the set of polynomials of degree less than or equal to dk, where d is a fixed positive integer. Then $X^{\mathcal{F}}$ is the d-uple embedding of $\mathbb{P}^n(\mathbb{K})$ in $\mathbb{P}^{m-1}(\mathbb{K})$, where $m = \binom{n+d}{n}$ = number of all monomials in n variables of degree at most d. In particular, for $n = 1$, $X^{\mathcal{F}}$ is the rational canonical curve of degree d in $\mathbb{P}^d(\mathbb{K})$. To see this, note that $F_k = (F_1)^k$ for all k, which means that $A^{\mathcal{F}} = \bigoplus_{i \geq 0} F_i$ is generated by F_1 as a $\mathbb{K}$-algebra. Therefore $\{(x^\alpha)_1 : |\alpha| \leq d\}$ is a set of $\mathbb{K}$-algebra generators of $A^{\mathcal{F}}$, where $|\alpha| := \alpha_1 + \ldots + \alpha_n$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_+)^n$. It follows by corollary 1.1.5 that $X^{\mathcal{F}}$ is the closure in $\mathbb{P}^{m-1}(\mathbb{K})$ of the image of $\mathbb{K}^n$ under the map given by all monomials in $x_1, \ldots, x_n$ of degree less than or equal to d. But the latter closure is identical with the image of $\mathbb{P}^n(\mathbb{K})$ under the map given by all monomials in $x_0, x_1 \ldots, x_n$ of degree equal to d (since they agree on a dense open set), and this is precisely the d-uple embedding of $\mathbb{P}^n$ [14, Exercise I.2.12].

Example 1.2.4. X is again $\mathbb{K}^n$ as above. Assume $n \geq 2$. Fix an integer k with $1 \leq k < n$. Let F_1 be the $\mathbb{K}$-linear span of all monomials of degree less than or equal to two excluding those of the form $x_i x_j$ with $i \geq j > k$. Let $F_d = (F_1)^d$ for $d \geq 1$. Then

$X^{\mathcal{F}}$ is isomorphic to the variety resulting from blowing up $\mathbb{P}^n(\mathbb{K})$ along the subspace $V := V(x_0, \ldots, x_k)$. To show this, let Y be the blow up of $\mathbb{P}^n(\mathbb{K})$ along V. Then variety Y is the closure in $\mathbb{P}^n(\mathbb{K}) \times \mathbb{P}^k(\mathbb{K})$ of the image of $\mathbb{K}^n$ under the map $(x_1, \ldots, x_n) \mapsto ([1 : x_1 : \cdots : x_n], [1 : x_1 : \cdots : x_k])$. Embedding $\mathbb{P}^n(\mathbb{K}) \times \mathbb{P}^k(\mathbb{K})$ into $\mathbb{P}^{kn+k+n}(\mathbb{K})$ via the Segre map, we see that Y is isomorphic to the closure in $\mathbb{P}^{kn+k+n}(\mathbb{K})$ of the image of X under the map

$$(x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n : x_1 : x_1^2 : \cdots : x_1x_n : \cdots : x_k : x_kx_1 : \cdots : x_kx_n] \,.$$

Composing with an automorphism of $\mathbb{P}^{kn+k+n}(\mathbb{K})$, we may assume that the map is $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n : x_1^2 : x_1x_2 : \cdots : x_1x_n : x_2^2 : x_2x_3 : \cdots : x_2x_n : \cdots : x_k^2 : x_kx_{k+1} : \cdots : x_kx_n : 0 : \cdots : 0]$. Projecting onto the first $N := (n+1)+n+\cdots+(n-k+1)$ coordinates, we see that Y is isomorphic to the closure in $\mathbb{P}^{N-1}(k)$ of the image of $\mathbb{K}^n$ under the map given by all monomials in F_1. But by corollary 1.1.5 this is precisely $X^{\mathcal{F}}$!

Example 1.2.5 (see [26, Example 1.2.5])**.** Let X be a normal affine variety with trivial divisor class group $\operatorname{Cl} X$ (e.g. $\mathbb{K}^n$, $(\mathbb{K}^*)^n$, or any X whose coordinate ring is a unique factorization domain), see section 2.0.2 for a brief introduction to class group and divisors. Let $\bar{X} \subseteq \mathbb{P}^L(\mathbb{K})$ be any normal projective completion of X. We now show that if $X_\infty := \bar{X} \setminus X$ is irreducible, then the embedding $X \hookrightarrow \bar{X}$ arises from a filtration. Without loss of generality we may assume that $\bar{X}$ does not lie in any proper hyperplane in $\mathbb{P}^L(\mathbb{K})$. Let the homogeneous coordinates of $\mathbb{P}^L(\mathbb{K})$ be $[x_0 : \ldots : x_L]$. Let D be the Cartier divisor on $\bar{X}$ corresponding to the hyperplane section by x_0, i.e. the equation for D on $\bar{X} \setminus V(x_i)$ is x_0/x_i. The Weil divisor associated to D is of the form $[D] = a_0[X_\infty] + \sum_{j=1}^m a_j[\bar{V}_j]$, where for each j, $\bar{V}_j$ is the closure in $\bar{X}$ of a codimension one subvariety V_j of X. Since $\operatorname{Cl} X = 0$, there is a rational function f on X such that the principal divisor of f on X is $[\operatorname{div}_X(f)] = \sum_{j=1}^m a_j[V_j]$. Let $[D'] := [D] - [\operatorname{div}_{\bar{X}}(f)]$, where $[\operatorname{div}_{\bar{X}}(f)]$ is the principal divisor of f on $\bar{X}$. But $[\operatorname{div}_{\bar{X}}(f)] = \sum_{j=1}^m a_j[\bar{V}_j] + a[X_\infty]$ for some $a \in \mathbb{Z}$. It follows that $[D'] = a'[X_\infty]$ for some $a' \in \mathbb{Z}$. But then $a' = \deg D' = \deg D = \deg \bar{X} > 0$ [14, Exercise

II.6.2], and therefore by the definition of the line bundle associated to a Cartier divisor [10, Subsection B4.4], $1 \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(D'))$. Recall that by our assumption $\bar{X}$ does not lie in any proper hyperplane in $\mathbb{P}^L(\mathbb{K})$. It follows that $\dim_{\mathbb{K}} \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(D)) = L + 1$, and hence $\dim_{\mathbb{K}} \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(D'))$ is also $L + 1$. Let $g_1, \ldots, g_L \in \mathbb{K}(X)$ such that $1, g_1, \ldots, g_L$ is a vector space basis of $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(D'))$. Since $(g_i) + D' \geq 0$ for all $i = 1, \ldots, L$, it follows that none of the g_i's has any pole on X, and hence each $g_i \in \mathbb{K}[X]$. Let $\phi : X \to \mathbb{K}^L$ be the closed immersion which sends $x \in X$ to $(g_1(x), \ldots, g_L(x))$. Then $\bar{X}$ is precisely the closure of $\phi(X)$ in $\mathbb{P}^L(\mathbb{K})$ and (by corollary 1.1.5) the embedding $X \hookrightarrow \bar{X} \subseteq \mathbb{P}^L(\mathbb{K})$ is induced by a filtration on $\mathbb{K}[X]$.

Example 1.2.6 (see [25, Remark 1] and [26, Example 1.2.6])**.** Not all projective completions of affine varieties are determined by a filtration. Our example below is a variation of an example by Mike Roth and Ravi Vakil considered in [33]. Let X' be a nonsingular cubic curve in $\mathbb{P}^2(\mathbb{K})$. Let O be one of its 9 inflection points. Recall that X' can be given the structure of an abelian group with O as the origin and that $\phi : P \to [P] - [O]$ gives an injective group homomorphism of X' into its class group $\operatorname{Cl} X'$ [14, Example II.6.10.2]. There are only countably many points $P \in X'$ such that P is a torsion point, i.e. $k \cdot P = 0$ for some $k > 0$ [34, Section III.3.4]. Pick any *non-torsion* point P of X'. Then $X := X' \setminus \{P\}$ is an affine variety [12, Proposition 5]. We claim that there is no homogeneous polynomial f in $\mathbb{K}[x_0, x_1, x_2]$ such that $V(f) \cap X' = \{P\}$. Indeed, assume to the contrary that there is a homogeneous polynomial f of degree $d > 0$ such that $V(f) \cap X' = \{P\}$. By Bezout's theorem $[\operatorname{div}(f)] = 3d[P]$, where $[\operatorname{div}(f)]$ is the divisor on X' associated to f. Let l be the equation of the tangent line of X' at O. Since O is an inflection point, it follows that $V(l) \cap X' = \{O\}$ and thus the divisor of l on X' is $[\operatorname{div}(l)] = 3[O]$. Then f/l^d is a rational function on X', and its corresponding divisor on X' is $[\operatorname{div}(f/l^d)] = 3d[P] - 3d[O]$. Since $[\operatorname{div}(f/l^d)]$ is a principal divisor, it follows that $[\operatorname{div}(f/l^d)] = 0 \in \operatorname{Cl} X'$. But then $\phi(3d \cdot P) = 3d[P] - 3d[O] = [\operatorname{div}(f/l^d)] = 0$, and hence $3d \cdot P = 0$, since ϕ is one-to-one. Therefore P is a torsion point, and we have a contradiction. Thus the claim is true and there is no homogeneous polynomial f such that $V(f) \cap X' = \{P\}$. It follows by proposition 1.1.2 that there is no integer $d > 0$ such that the embedding of X into the image of the d-uple embedding of X' is induced by a filtration!

1.3 Preserving Intersections at Infinity

Let X be an affine variety over $\mathbb{K}$. Recall that for subsets $V_1, \ldots, V_m$ of X, a completion $\psi : X \hookrightarrow Z$ is said to *preserve the intersection of* $V_1, \ldots, V_m$ *at* ∞ if $\overline{V}_1 \cap \cdots \cap \overline{V}_m \cap X_\infty = \emptyset$, where $X_\infty := Z \setminus X$ is the set of 'points at infinity' and $\overline{V}_j$ is the closure of V_j in Z for every j.

Lemma 1.3.1. *Let $\mathcal{F} = \{F_d : d \geq 0\}$ be a complete filtration on $A := \mathbb{K}[X]$, and $\psi_{\mathcal{F}} : X \hookrightarrow X^{\mathcal{F}} := \operatorname{Proj} A^{\mathcal{F}}$ be the corresponding completion.*

1. *For each ideal $\mathfrak{q}$ of A, let $\mathfrak{q}^{\mathcal{F}} := \bigoplus_{d \geq 0}(\mathfrak{q} \cap F_d) \subseteq A^{\mathcal{F}}$. Then the closure of $V(\mathfrak{q}) \subseteq X$ in $X^{\mathcal{F}}$ is $V(\mathfrak{q}^{\mathcal{F}})$.*
2. *Let $V_1, \ldots, V_m$ be Zariski closed subsets of X with $V_i = V(\mathfrak{q}_i)$ for ideals $\mathfrak{q}_i \subseteq A$ for each i. Let $\mathcal{I}$ be the ideal of $A^{\mathcal{F}}$ generated by $\mathfrak{q}_1^{\mathcal{F}}, \ldots, \mathfrak{q}_m^{\mathcal{F}}$ and $(1)_1$. Then $\psi_{\mathcal{F}}$ preserves the intersection of $V_1, \ldots, V_m$ at ∞ iff the $\sqrt{\mathcal{I}} \supseteq A_+^{\mathcal{F}}$, where $A_+^{\mathcal{F}} := \bigoplus_{d>0} F_d$ is the* irrelevant *ideal of $A^{\mathcal{F}}$.*

Proof. 1. Recall (example 1.0.6) that there exists $d > 0$ such that $(A^{\mathcal{F}})^{[d]} := \bigoplus_{k \geq 0} F_{kd}$ is generated by F_d as a $\mathbb{K}$-algebra. Define a new filtration $\mathcal{G} := \{G_k : k \geq 0\}$ on A by $G_k := F_{kd}$. Let $\{1, g_1, \ldots, g_m\}$ be a $\mathbb{K}$-vector space basis of G_1. Then $A^{\mathcal{G}} \cong (A^{\mathcal{F}})^{[d]}$ and by corollary 1.1.5, $X^{\mathcal{G}} := \operatorname{Proj} A^{\mathcal{G}}$ is the closure in $\mathbb{P}^m(\mathbb{K})$ of $\phi(X)$, where $\phi : X \to \mathbb{K}^m$ is defined by: $\phi(x) := (g_1(x), \ldots, g_m(x))$.

Let $\mathfrak{q}$ be an ideal of A and $V := V(\mathfrak{q})$ be the Zariski closed subset of X defined by $\mathfrak{q}$. Let $\mathfrak{p} := \ker \phi^*$ and $\mathfrak{r} := (\phi^*)^{-1}(\mathfrak{q})$, where $\phi^* : \mathbb{K}[y_1, \ldots, y_m] \to A$ is the pull back by means of ϕ. Identify X with $V(\mathfrak{p})$ and V with $V(\mathfrak{r})$ in $\mathbb{K}^m$. Then $X^{\mathcal{G}}$ and the closure $\overline{V}^{\mathcal{G}}$

of V in $X^{\mathcal{G}}$ are the Zariski closed subsets of $\mathbb{P}^m(\mathbb{K})$ determined by the *homogenizations* $\tilde{\mathfrak{p}}$ of $\mathfrak{p}$ and, respectively, $\tilde{\mathfrak{r}}$ of $\mathfrak{r}$ with respect to y_0.

Moreover, the closed embedding $\Phi : X^{\mathcal{G}} \hookrightarrow \mathbb{P}^m(\mathbb{K})$ is induced by the surjective homomorphism $\Phi^* : \mathbb{K}[y_0, \ldots, y_m] \to A^{\mathcal{G}}$ which maps $y_0 \mapsto (1)_1$ and $y_i \mapsto (g_i)_1$ for $1 \leq i \leq m$. Therefore $\overline{V}^{\mathcal{G}}$ in $X^{\mathcal{G}}$ is defined by the ideal $\Phi^*(\tilde{\mathfrak{r}})$. But the d-th graded component of $\Phi^*(\tilde{\mathfrak{r}})$ is

$$\begin{aligned}
\Phi^*((\tilde{\mathfrak{r}})_d) &:= \{\Phi^*(\tilde{f}(y_0, \ldots, y_m)) : \tilde{f} \in \tilde{\mathfrak{r}},\ \tilde{f} \text{ homogeneous},\ \deg(\tilde{f}) = d\} \\
&= \{\tilde{f}((1)_1, (g_1)_1, \ldots, (g_m)_1) : \\
&\qquad \tilde{f} \text{ homogeneous in } y_0, \ldots, y_m,\ \deg(\tilde{f}) = d,\ \tilde{f}(1, y_1, \ldots, y_m) \in \mathfrak{r}\} \\
&= \{(\tilde{f}(1, g_1, \ldots, g_m))_d : \\
&\qquad \tilde{f} \text{ homogeneous in } y_0, \ldots, y_m,\ \deg(\tilde{f}) = d, \tilde{f}(1, g_1, \ldots, g_m) \in \mathfrak{q}\} \\
&= \{(f(g_1, \ldots, g_m))_d : \\
&\qquad f \text{ polynomial in } y_1, \ldots, y_m,\ \deg(f) \leq d,\ f(g_1, \ldots, g_m) \in \mathfrak{q}\} \\
&= \{(g)_d : g \in \mathfrak{q} \cap G_d\},
\end{aligned}$$

where the last equality is a consequence of $A^{\mathcal{G}} = \mathbb{K}[(1)_1, (g_1)_1, \ldots, (g_m)_1]$. Then $\Phi^*(\tilde{\mathfrak{r}}) = \bigoplus_{d \geq 0} \Phi^*((\tilde{\mathfrak{r}})_d) = \bigoplus_{d \geq 0} \{(g)_d : g \in \mathfrak{q} \cap G_d\} = \mathfrak{q}^{\mathcal{G}}$. Now recall (example 1.0.5) that $X^{\mathcal{F}}$ and $X^{\mathcal{G}}$ are isomorphic and this isomorphism is induced by the inclusion $A^{\mathcal{G}} \subseteq A^{\mathcal{F}}$. Since $\mathfrak{q}^{\mathcal{F}} \cap A^{\mathcal{G}} = \mathfrak{q}^{\mathcal{G}}$, it follows that the Zariski closed subset of $X^{\mathcal{G}}$ determined by $\mathfrak{q}^{\mathcal{G}}$ is isomorphic to the Zariski closed subset of $X^{\mathcal{F}}$ determined by $\mathfrak{q}^{\mathcal{F}}$.

2. $\psi_{\mathcal{F}}$ preserves the intersection of $V_1, \ldots, V_m$ at ∞ iff $\overline{V}_1 \cap \cdots \cap \overline{V}_m \cap X_\infty = \emptyset$. By part 1 $\overline{V}_j = V(\mathfrak{q}_j^{\mathcal{F}})$ for each j, and by theorem 1.1.4 $X_\infty = V((1)_1)$. Therefore $\overline{V}_1 \cap \cdots \cap \overline{V}_m \cap X_\infty$ is determined by the ideal of $A^{\mathcal{F}}$ generated by $(1)_1, \mathfrak{q}_1^{\mathcal{F}}, \ldots, \mathfrak{q}_m^{\mathcal{F}}$, which is precisely the definition of $\mathcal{I}$. Then the projective version of Nullstellensatz (see section 1.0.2) implies $V(\mathcal{I}) = \emptyset$ iff $A_+^{\mathcal{F}} \subseteq \sqrt{\mathcal{I}}$. □

Theorem 1.3.2 (see [25, Theorem 1.2(1)] and [26, Theorem 1.3.1])**.** *Let $V_1, \ldots, V_m$ be Zariski closed subsets in an affine variety X such that $\cap_{i=1}^m V_i$ is a finite set. Then there* *is a complete filtration $\mathcal{F}$ on $\mathbb{K}[X]$ such that $\psi_{\mathcal{F}}$ preserves the intersection of the V_i's at ∞.*

Proof. Let $X \subseteq \mathbb{K}^n$ and the ideals in $\mathbb{K}[x_1, \ldots, x_n]$ defining $X, V_1, \ldots, V_m$ be respectively $\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_m$ with $\mathfrak{q}_j \supseteq \mathfrak{p}$ for each j.

Claim. *For each $i = 1, \ldots, n$, there is an integer $d_i \geq 1$ such that*

$$x_i^{d_i} = f_{i,1} + \ldots + f_{i,m} + g_i \tag{1.2}$$

for some $f_{i,j} \in \mathfrak{q}_j$ and a polynomial $g_i \in \mathbb{K}[x_i]$ of degree less that d_i.

Proof. If $V_1 \cap \ldots \cap V_m = \emptyset$, then by Nullstellensatz $\langle \mathfrak{q}_1, \ldots, \mathfrak{q}_m \rangle$ is the unit ideal in $\mathbb{K}[x_1, \ldots, x_n]$, and the claim is trivially satisfied with $g_i := 0$ for each i. So assume

$$V_1 \cap \ldots \cap V_m = \{P_1, \ldots, P_k\} \subseteq \mathbb{K}^n,$$

for some $k \geq 1$. Let $P_i = (a_{i,1}, \ldots, a_{i,n}) \in \mathbb{K}^n$. For each $i = 1, \ldots, n$, let

$$h_i := (x_i - a_{1,i})(x_i - a_{2,i}) \cdots (x_i - a_{k,i}).$$

By Nullstellensatz, for some $d'_i \geq 1$, $h_i^{d'_i} \in \langle \mathfrak{q}_1, \ldots, \mathfrak{q}_m \rangle$, i.e. $h_i^{d'_i} = f_{i,1} + \ldots + f_{i,m}$ for some $f_{i,j} \in \mathfrak{q}_j$. Substituting $h_i = \prod_j (x_i - a_{j,i})$ in the preceding equation we see that the claim holds with $d_i := kd'_i$. □

Below for $S \subseteq \mathbb{K}[X]$ we denote by $\mathbb{K}\langle S \rangle$ the $\mathbb{K}$-linear span of S, and for an element $g \in \mathbb{K}[x_1, \ldots, x_n]$, we denote by $\bar{g}$ the image of g in $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n]/\mathfrak{p}$. Fix a set of $f_{i,j}$'s satisfying the conclusion of the previous claim. Then define a filtration $\mathcal{F}$ on $\mathbb{K}[X]$ as follows: let

$$F_0 := \mathbb{K},$$

$$F_1 := \mathbb{K}\langle 1, \bar{x}_1, \ldots, \bar{x}_n, \bar{f}_{1,1}, \ldots, \bar{f}_{n,m} \rangle,$$

$$F_k := F_1^k \text{ for } k > 1,$$

$$\mathcal{F} := \{F_i : \ i \geq 0\}.$$

Clearly $\mathcal{F}$ is a complete filtration. We now show that this $\mathcal{F}$ satisfies the conclusion of the theorem. By lemma 1.3.1 this is equivalent to showing that $\sqrt{\mathcal{I}} \supseteq \mathbb{K}[X]_+^{\mathcal{F}}$, where $\mathcal{I}$ is the ideal generated by $\bar{\mathfrak{q}}_1^{\mathcal{F}}, \ldots, \bar{\mathfrak{q}}_m^{\mathcal{F}}$ and $(1)_1$ in $\mathbb{K}[X]^{\mathcal{F}}$.

From the construction of $\mathcal{F}$ it follows that $\mathbb{K}[X]_+^{\mathcal{F}}$ is generated by the elements $(1)_1, (\bar{x}_1)_1, \ldots, (\bar{x}_n)_1, (\bar{f}_{1,1})_1, \ldots, (\bar{f}_{n,m})_1$. Note that $\bar{f}_{i,j} \in \bar{\mathfrak{q}}_j$ for each i, j, so that $(\bar{f}_{i,j})_1 \in \bar{\mathfrak{q}}_j^{\mathcal{F}} \subseteq \mathcal{I}$. Moreover, $(1)_1 \in \mathcal{I}$. So, all we really need to show is that $(\bar{x}_i)_1 \in \sqrt{\mathcal{I}}$ for all $i = 1, \ldots, n$.

Reducing equation (1.2) mod $\mathfrak{p}$, we have $(\bar{x}_i)^{d_i} = \bar{f}_{i,1} + \ldots + \bar{f}_{i,m} + \bar{g}_i \in \mathbb{K}[X]$ for all $i = 1, \ldots, n$. Let $g_i = \sum_{j=0}^{d_i-1} a_{i,j} x_i^j$. Then in $\mathbb{K}[X]^{\mathcal{F}}$,

$$((\bar{x}_i)_1)^{d_i} = ((1)_1)^{d_i-1}((\bar{f}_{i,1})_1 + \ldots + (\bar{f}_{i,m})_1) + \sum_{j=0}^{d_i-1} a_{i,j}((\bar{x}_i)_1)^j((1)_1)^{d_i-j}.$$

All of the summands in the right hand side lie inside $\mathcal{I}$, hence $((\bar{x}_i)_1)^{d_i} \in \mathcal{I}$ for all $i = 1, \ldots, n$, as required. □

Recall that given a polynomial map $f = (f_1, \ldots, f_q) : X \to \mathbb{K}^q$, $a = (a_1, \ldots, a_q) \in \mathbb{K}^q$ and a completion ψ of X, ψ *preserves* $\{f_1, \ldots, f_n\}$ *at* ∞ *over* a if ψ preserves the intersection of the hypersurfaces $H_i(a) := \{x \in X : f_i(x) = a_i\}$, $i = 1, \ldots, q$.

Example 1.3.3. Consider map $f : \mathbb{K}^2 \to \mathbb{K}^2$ given by $f(x, y) := (x, y + x^3)$. For $a := (a_1, a_2) \in \mathbb{K}^2$,

$$H_1(a) = \{(a_1, y) : y \in \mathbb{K}\},$$
$$H_2(a) = \{(x, a_2 - x^3) : x \in \mathbb{K}\}.$$

We claim that in the usual completion $\mathbb{P}^2(\mathbb{K})$ of $\mathbb{K}^2$, the closures of $H_1(a)$ and $H_2(a)$ intersect at a point P at infinity for each $a \in \mathbb{K}^2$, and hence $\mathbb{P}^2(\mathbb{K})$, as the natural completion of $\mathbb{K}^2$, does not preserve $\{f_1, \ldots, f_n\}$ at ∞ over any point of $\mathbb{K}^2$.

Indeed, write the homogeneous coordinates of $\mathbb{P}^2(\mathbb{K})$ as $[z : x : y]$ and identify $\mathbb{K}^2$ with $\mathbb{P}^2(\mathbb{K}) \setminus V(z)$. Let $a \in \mathbb{K}^2$. When $\mathbb{K} = \mathbb{C}$, the 'infinite' points in $\overline{H}_i(a)$ can be described as the limits of points in $H_i(a)$. Therefore, the points at infinity of $H_1(a)$ are $\lim_{|y|\to\infty}[1 : a_1 : y] = \lim_{|y|\to\infty}[1/y : a_1/y : 1] = [0 : 0 : 1]$. Similarly, the infinite part of $H_2(a)$ is $\lim_{|x|\to\infty}[1 : x : a_2 - x^3] = \lim_{|x|\to\infty}[1/(a_2 - x^3) : x/(a_2 - x^3) : 1] = [0 : 0 : 1]$, and hence the claim is true with $P := [0 : 0 : 1]$.

To verify the claim for an arbitrary $\mathbb{K}$, one has to apply lemma 1.3.1 with $X = \mathbb{K}^2$ and calculate $\overline{H}_i(a) \cap X_\infty = V(\mathfrak{q}_i^{\mathcal{F}}(a), (1)_1)$, where $\mathfrak{q}_i(a)$ is the ideal of $H_i(a)$. A straightforward calculation shows: the graded ring $\mathbb{K}[X]^{\mathcal{F}}$ corresponding to the embedding $\mathbb{K}^2 \hookrightarrow \mathbb{P}^2(\mathbb{K})$ is isomorphic to $\mathbb{K}[x, y, z]$ where z plays the role of $(1)_1$, and $\mathfrak{q}_i^{\mathcal{F}}(a)$ is the homogenization of $\mathfrak{q}_i(a)$ with respect to z. Then $\mathfrak{q}_1(a) = \langle x - a_1\rangle$ and $\mathfrak{q}_2(a) = \langle y + x^3 - a_2\rangle$, so that $\mathfrak{q}_1^{\mathcal{F}}(a) = \langle x - a_1 z\rangle$ and $\mathfrak{q}_2^{\mathcal{F}}(a) = \langle yz^2 + x^3 - a_2 z^3\rangle$. Therefore $\overline{H}_1(a) \cap X_\infty = V(x - a_1 z, z) = V(x, z) = \{[0 : 0 : 1]\}$. Similarly $\overline{H}_2(a) \cap X_\infty = V(yz^2 + x^3 - a_2 z, z) = V(x, z) = \{[0 : 0 : 1]\}$ and the claim is valid with the same P as in $\mathbb{K} = \mathbb{C}$ case.

Because it is simpler to describe, we will from now on frequently use only the limit argument (valid only for $\mathbb{K} = \mathbb{C}$) in order to find the points at infinity of various subvarieties of a given X. In all these cases, the analogous results also follow over an arbitrary algebraically closed field $\mathbb{K}$ by means of straightforward calculations (and if $\operatorname{char} \mathbb{K} = 0$, by Tarski-Lefschetz principle).

We now find, following the proof of Theorem 1.3.2, a completion of $\mathbb{K}^2$ which preserves $\{f_1, \ldots, f_n\}$ at ∞ over 0. In the notation of theorem 1.3.2, $\mathfrak{q}_1 = \langle x\rangle$ and $\mathfrak{q}_2 = \langle y + x^3\rangle$. Observe that $x \in \mathfrak{q}_1$, and y satisfies

$$y = -x^3 + (y + x^3),$$

with $x^3 \in \mathfrak{q}_1$ and $y + x^3 \in \mathfrak{q}_2$. Let filtration $\mathcal{F} := \{F_i : i \geq 0\}$ on $\mathbb{K}[x, y]$ be defined as follows: $F_0 := \mathbb{K}$, $F_1 := \mathbb{K}\langle 1, x, y, x^3\rangle$, and $F_k := (F_1)^k$ for $k > 1$. Then as in the proof of theorem 1.3.2, completion $\psi_{\mathcal{F}}$ preserves $\{f_1, \ldots, f_n\}$ at ∞ over 0. Let us now show this directly. By corollary 1.1.5, the corresponding completion $X^{\mathcal{F}}$ is isomorphic to the closure in $\mathbb{P}^3(\mathbb{K})$ of the image of $\phi : \mathbb{K}^2 \to \mathbb{P}^3(\mathbb{K})$, where $\phi(x, y) := [1 : x : y : x^3]$.

Then $\phi(H_1(a)) = \{[1 : a_1 : y : a_1^3] : y \in \mathbb{K}\}$ and limit $\lim_{y\to\infty}[1 : a_1 : y : a_1^3] = \lim_{y\to\infty}[1/y : a_1/y : 1 : a_1^3/y] = [0 : 0 : 1 : 0]$, for $a \in \mathbb{K}^2$, so that the only point at infinity of $\overline{H}_1(a)$ is $[0 : 0 : 1 : 0]$. Similarly, $\phi(H_1(a)) = \{[1 : x : a_2 - x^3 : x^3] : x \in \mathbb{K}\}$ and $\lim_{x\to\infty}[1 : x : a_2 - x^3 : x^3] = \lim_{x\to\infty}[1/x^3 : 1/x^2 : (a_2 - x^3)/x^3 : 1] = [0 : 0 : -1 : 1]$. Therefore $\overline{H}_2(a)$ also has only one point at infinity and it is $[0 : 0 : -1 : 1]$. It follows that $\overline{H}_1(a) \cap \overline{H}_2(a) \cap X_\infty = \emptyset$ for all a, i.e. $X^{\mathcal{F}}$ preserves $\{f_1, \ldots, f_n\}$ at ∞ over *every* point of $\mathbb{K}^2$.

Example 1.3.4. Let $f(x, y) := (x, y)$ on $\mathbb{K}^2$. Then for each $a = (a_1, a_2) \in \mathbb{K}^2$, $H_1(a) = \{(a_1, y) : y \in \mathbb{K}\}$ and $H_2(a) = \{(x, a_2) : x \in \mathbb{K}\}$. Consider filtration $\mathcal{F}$ on $\mathbb{K}[x, y]$ defined by: $F_0 := \mathbb{K}$, $F_1 := \mathbb{K}\langle 1, x, y, xy, x^2y^2\rangle$, and $F_k := (F_1)^k$ for $k \geq 2$. By corollary 1.1.5, $X^{\mathcal{F}}$ is the closure of the image of $\mathbb{K}^2$ under the map $\phi : \mathbb{K}^2 \hookrightarrow \mathbb{P}^4(\mathbb{K})$ defined by: $\phi(x, y) = [1 : x : y : xy : x^2y^2]$. Then $\phi(H_1(a)) = \{[1 : a_1 : y : a_1y : a_1^2y^2] : y \in \mathbb{K}\}$. If $a_1 = 0$, then $\phi(H_1(a)) = \{[1 : 0 : y : 0 : 0] : y \in \mathbb{K}\}$, and hence the only point at infinity in $\overline{\phi(H_1(a))}$ is $[0 : 0 : 1 : 0 : 0]$. But if $a_1 \neq 0$, then dividing all coordinates by $a_1^2y^2$, we see that the point at infinity in $\overline{\phi(H_1(a))}$ is $[0 : 0 : 0 : 0 : 1]$. Similarly, $\phi(H_2(a)) = \{[1 : x : a_2 : y : a_2x : a_2^2x^2] : x \in \mathbb{K}\}$ and the only point at infinity in $\overline{\phi(H_2(a))}$ is $[0 : 1 : 0 : 0 : 0]$ if $a_2 = 0$, and $[0 : 0 : 0 : 0 : 1]$ if $a_2 \neq 0$. Therefore $X^{\mathcal{F}}$ preserves $\{f_1, \ldots, f_n\}$ at ∞ over a iff $a_1 = 0$ or $a_2 = 0$, i.e. iff a belongs to the union of the coordinate axes.

Let $f : X \to Y$ be a generically finite map of affine varieties of the same dimension. Given any $y \in Y$ such that $f^{-1}(y)$ is finite, theorem 1.3.2 guarantees the existence of a projective completion of X that preserves $\{f_1, \ldots, f_n\}$ at ∞ over y. But as the preceding example shows, the completion might fail to preserve $\{f_1, \ldots, f_n\}$ at ∞ over 'most of the' points in the image of f. This suggests that we should look for a completion ψ which preserves $\{f_1, \ldots, f_n\}$ at ∞ over y for *generic* $y \in Y$, i.e. *ψ preserves $\{f_1, \ldots, f_n\}$ at ∞.* We will demonstrate two ways to accomplish this goal - we start with a simpler-to-prove theorem 1.3.5 and will present a stronger version in theorem 1.3.7 following (cf. [25, Theorem 1.2] and [26, Theorem 1.3.4]).

Theorem 1.3.5. [*] *Let $f : X \to Y \subseteq \mathbb{K}^q$ be a generically finite map of affine varieties of same dimension. Include $\mathbb{K}^q$ into $(\mathbb{P}^1(\mathbb{K}))^q$ via the componentwise inclusion $(a_1, \ldots, a_q) \mapsto ([1 : a_1], \ldots, [1 : a_q])$. Let $\phi : X \hookrightarrow Z$ be any completion of X. Define $\bar{X}$ to be the closure of the graph of f in $Z \times (\mathbb{P}^1(\mathbb{K}))^q$. Then $\bar{X}$ preserves $\{f_1, \ldots, f_n\}$ at ∞. If ϕ comes from some filtration on $\mathbb{K}[X]$, then there is a filtration $\mathcal{F}$ on $\mathbb{K}[X]$ and a commutating diagram as follows:*

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ \bar{X} & \xrightarrow{\cong} & X^{\mathcal{F}} \end{array}$$

Proof. Let $\pi := (\pi_1, \ldots, \pi_q) : Z \times (\mathbb{P}^1(\mathbb{K}))^q \to (\mathbb{P}^1(\mathbb{K}))^q$ be the natural projection. Then π maps $\bar{X}$ onto the closure $\bar{Y}$ of Y in $(\mathbb{P}^1(\mathbb{K}))^q$. Let $V := Z \setminus X$. Then $\tilde{V} := (V \times (\mathbb{P}^1(\mathbb{K}))^q) \cap \bar{X}$ is a proper Zariski closed subset of $\bar{X}$. Since Z is complete, it follows that $\pi(\tilde{V})$ is a proper Zariski closed subset of $\bar{Y}$. We now show that for all $y \in Y \setminus \pi(\tilde{V})$, $\bar{X}$ preserves $\{f_1, \ldots, f_n\}$ at ∞ over y.

Pick an arbitrary $y := (y_1, \ldots, y_q) \in Y$ such that $\bar{X}$ does not preserve $\{f_1, \ldots, f_n\}$ at ∞ over y. It suffices to show that $y \in \pi(\tilde{V})$. As usual, let $H_i(y) := \{x \in X : f_i(x) = y_i\}$. By assumption there is $\tilde{x} \in \bar{H}_1(y) \cap \cdots \cap \bar{H}_q(y) \cap (\bar{X} \setminus X)$, where for each k, $\bar{H}_k(y)$ is the closure of $H_k(y)$ in $\bar{X}$. Fix a k, $1 \le k \le q$. Note that $\pi_k(H_k(y)) = \{y_k\}$. By continuity of π_k it follows that $\pi_k(\bar{H}_k(y)) = \{y_k\}$. But then $\pi_k(\tilde{x}) = y_k$. It follows that $\pi(\tilde{x}) = y$ and hence $\tilde{x}$ is of the form (z, y) for some $z \in Z$. We claim that z does not lie in X. Indeed, if $z \in X$, it would imply $\tilde{x} \in (X \times Y) \cap (\bar{X} \setminus \psi(X))$, where $\psi : X \hookrightarrow \bar{X}$ is the inclusion. Consider the chain of inclusions: $\psi(X) \subseteq X \times Y \subseteq Z \times (\mathbb{P}^1(\mathbb{K}))^q$. Note:

1. $\psi(X)$ is the graph of f in $X \times Y$, and hence is a Zariski closed subvariety of $X \times Y$.
2. $X \times Y$ is Zariski *open* in $Z \times (\mathbb{P}^1(\mathbb{K}))^q$.
3. If $T \subseteq U \subseteq W$ are topological spaces such that T is closed in U and U is open in

[*]The idea of looking at the construction of theorem 1.3.5 is due to Professor A. Khovanskii.

W, then $\bar{T} \cap U = T$, where $\bar{T}$ is the closure of T in W.

Since $\bar{X}$ is by definition the closure of $\psi(X)$ in $Z \times (\mathbb{P}^1(\mathbb{K}))^q$, it follows via the above observations that $\bar{X} \cap (X \times Y) = \psi(X)$, so that $(X \times Y) \cap (\bar{X} \setminus \psi(X)) = \emptyset$. This contradiction proves the claim. It follows that $z \in Z \setminus X = V$. Then $\tilde{x} \in \tilde{V}$. Therefore $y = \pi(\tilde{x}) \in \pi(\tilde{V})$ and the first claim of the theorem is proved.

As for the last claim, note that if ϕ comes from a filtration, then by corollary 1.1.5 we may assume that $Z \subseteq \mathbb{P}^p(\mathbb{K})$ for some p and $\phi(x) = [1 : g_1(x) : \cdots : g_p(x)]$ for some $g_1, \ldots, g_p \in \mathbb{K}[X]$. Hence the inclusion $\psi : X \hookrightarrow \bar{X}$ is of the form: $\psi(x) = ([1 : g_1(x) : \cdots : g_p(x)], [1 : f_1(x)], \ldots, [1 : f_q(x)])$. Let $l := (p+1)2^q - 1$ and let us embed $\mathbb{P}^p(\mathbb{K}) \times (\mathbb{P}^1(\mathbb{K}))^q \hookrightarrow \mathbb{P}^l(\mathbb{K})$ via the *Segre* embedding s which maps $w := ([w_0 : \cdots : w_p], [w_{1,0} : w_{1,1}], \ldots, [w_{q,0} : w_{q,1}])$ to the point $s(w)$ whose homogeneous coordinates are monomials of degree $q+1$ in w of the form $w_i w_{1,j_1} w_{2,j_2} \cdots w_{q,j_q}$ where $0 \leq i \leq p$ and $0 \leq j_k \leq 1$ for each k. The component of $s \circ \psi$ corresponding to $i = j_1 = \cdots = j_q = 0$ is 1 and hence $s \circ \psi$ maps $x \in X$ to a point with homogeneous coordinates $[1 : h_1(x) : \cdots : h_l(x)]$ for some $h_1, \ldots, h_l \in \mathbb{K}[X]$. Then corollary 1.1.5 implies that there is a filtration $\mathcal{F}$ on $\mathbb{K}[X]$ such that the closure $\bar{X}$ of $s \circ \psi(X)$ in $\mathbb{P}^l(\mathbb{K})$ is isomorphic to $X^{\mathcal{F}}$ via an isomorphism which is identity on X. Morphism s being an isomorphism completes the proof. $\square$

Remark 1.3.6. Let $f : X \to Y$ be a map of n-dimensional affine varieties with generically finite fibers and ψ be a completion of X. Define $S_\psi := \{a \in f(X) : \psi$ preserves $\{f_1, \ldots, f_n\}$ at ∞ over $y\}$. It will be interesting to know if S_ψ has any intrinsic structure. In example 1.3.4 S_ψ was the union of two coordinate axes in $\mathbb{K}^2$, and hence a proper closed subset of $f(X)$. On the other hand, theorem 1.3.5 shows that there are completions ψ of X such that S_ψ contains a dense open subset of $f(X)$. We now give an example where S_ψ is indeed a proper dense open subset of $f(X)$, namely:

Let $X = Y = \mathbb{C}^2$ and $f : X \to Y$ be the map defined by $f_1 := x_1^3 + x_1^2 x_2 + x_1 x_2^2 - x_2$ and $f_2 := x_1^3 + 2x_1^2 x_2 + x_1 x_2^2 - x_2$. It is easy to see that f is quasifinite. Let $\phi : X \hookrightarrow \mathbb{P}^2(\mathbb{C})$

be the usual completion, and let $\psi : X \hookrightarrow \bar{X}$ be as in theorem 1.3.4. Let the coordinates of $\mathbb{P}^2(\mathbb{C})$ be $[Z : X_1 : X_2]$. Identify X with $\mathbb{P}^2(\mathbb{C}) \setminus V(Z)$, so that $x_i = X_i/Z$ for $i = 1, 2$. Then $X \ni (x_1, x_2) \stackrel{\psi}{\mapsto} ([1 : x_1 : x_2], [1 : f_1(x)], [1 : f_2(x)])$. We claim that $f(X) \setminus S_\psi$ is the line $L := \{(c, c) : c \in \mathbb{C}\}$.

Indeed, let $a := (a_1, a_2) \in Y$. Define, as usual, $H_i(a) := \{x \in X : f_i(x) = a_i\}$ for $i = 1, 2$. Let $C_i(a)$ be the closure in $\mathbb{P}^2(\mathbb{C})$ of $H_i(a)$ for each i. It is easy to see that $P := [0 : 0 : 1] \in C_1(a) \cap C_2(a)$. Choose local coordinates $\xi_1 := X_1/X_2$ and $\xi_2 := Z/X_2$ of $\mathbb{P}^2(\mathbb{C})$ near $P := [0 : 0 : 1]$. Equations of $f_{i,a}$ in (ξ_1, ξ_2) coordinates are:

$$\begin{aligned} f_{1,a} &= \xi_1^3 + \xi_1^2 + \xi_1 - \xi_2^2 - a_1\xi_2^3 \\ f_{2,a} &= \xi_1^3 + 2\xi_1^2 + \xi_1 - \xi_2^2 - a_2\xi_2^3 \end{aligned} \tag{1.3}$$

It follows that for each i, $C_i(a)$ is smooth at P (in particular, each has only one branch at P) and both admit parametrizations at P of the form

$$\gamma_{i,a}(t) := [t : t^2 + o(t^3) : 1], \tag{1.4}$$

where $o(t^3)$ means terms of order t^3 and higher. In (x_1, x_2) coordinates the parametrizations are of the form: $(x_1(t), x_2(t)) = (t + o(t^2), 1/t)$ for $t \neq 0$. Since $f_2(x) = f_1(x) + x_1^2 x_2$, it follows that for $t \neq 0$,

$$\begin{aligned} \psi(\gamma_{1,a}(t)) &= (\gamma_{1,a}(t), [1 : a_1], [1 : a_1 + \frac{(t + o(t^2))^2}{t}]) \\ &= (\gamma_{1,a}(t), [1 : a_1], [1 : a_1 + t + o(t^2)]) \end{aligned}$$

Therefore $\lim_{t\to 0} \psi(\gamma_{1,a}(t)) = (P, [1 : a_1], [1 : a_1])$. Since $f_1(x) = f_2(x) - x_1^2 x_2$, the same argument also gives $\lim_{t\to 0} \psi(\gamma_{2,a}(t)) = (P, [1 : a_2], [1 : a_2])$.

To summarize, we proved that if $a \in L$, then $(P, [1 : a_1], [1 : a_1]) \in \bar{H}_1(a) \cap \bar{H}_2(a) \cap X_\infty$, where as usual $\bar{H}_i(a)$ is the closure of $H_i(a)$ in $\bar{X}$ and $X_\infty := \bar{X} \setminus X$. It follows that $L \subseteq f(X) \setminus S_\psi$.

To prove the other inclusion, assume $a \in f(X) \setminus S_\psi$. Pick $z \in \bar{H}_1(a) \cap \bar{H}_2(a) \cap X_\infty$. Then $z = (Q, [1 : a_1], [1 : a_2])$ for a point $Q \in \mathbb{P}^2(\mathbb{C})$. Therefore it follows that $Q \in$

$(C_1(a) \setminus H_1(a)) \cap (C_2(a) \setminus H_2(a))$, where curves $C_i(a)$ are the closures in $\mathbb{P}^2(\mathbb{C})$ of $H_i(a)$, $i = 1, 2$. But the only possible choice for such point Q is point P. Therefore limit $\lim_{t\to 0} \psi(\gamma_{1,a}(t)) = z = \lim_{t\to 0} \psi(\gamma_{2,a}(t))$, which implies that $(P, [1 : a_1], [1 : a_1]) = (P, [1 : a_2], [1 : a_2])$. Therefore $a_1 = a_2$ and $a \in L$, as claimed.

Let $f : X \to Y \subseteq \mathbb{K}^q$ be as in theorem 1.3.5. In the theorem following we find completions with even stronger preservation property at ∞, namely completions that *preserve map f at* ∞ (remark-definition 0.1.2).

Theorem 1.3.7 (cf. [25, Theorem 1.2(2)] and [26, Theorem 1.3.4])**.** *Let $f : X \to Y \subseteq \mathbb{K}^q$ be a generically finite map of affine varieties of the same dimension. Then there is a complete filtration $\mathcal{F}$ on the coordinate ring of X such that $\psi_{\mathcal{F}}$ preserves map f at ∞.*

Proof. Choose a set of coordinates $x_1, \ldots, x_p$ (resp. $y_1, \ldots, y_q$) of X (resp. Y). Since f is generically finite, it follows that the coordinate ring of X is algebraic over the pullback of the coordinate ring of Y. In particular, each x_i satisfies a polynomial of the form:

$$\sum_{j=0}^{k_i} g_{i,j}(y)(x_i)^j = 0 \tag{1.5}$$

for some $k_i \geq 1$ and regular functions $g_{i,j}(y)$ on Y such that $g_{i,k_i} \neq 0 \in \mathbb{K}[Y]$. In abuse of notation, but for the sake of convenience, we implicitly identified in (1.5) variables y_k with polynomials f_k for each k. We continue to do so throughout this proof. Let $g_{i,j}(y) = \sum_\alpha c_{i,j,\alpha} y^\alpha$ be an arbitrary representation of $g_{i,j}$ in $\mathbb{K}[Y]$. For each i, j with $1 \leq i \leq p$ and $0 \leq j \leq k_i$, let $d_{i,j} := \deg_y(\sum_\alpha c_{i,j,\alpha} y^\alpha) := \max\{|\alpha| : c_{i,j,\alpha} \neq 0\}$, where $\alpha = (\alpha_1, \ldots, \alpha_q) \in (\mathbb{Z}_+)^q$ and $|\alpha| := \alpha_1 + \cdots + \alpha_q$. Let $d_0 := \max\{d_{i,0} : 1 \leq i \leq p\}$ and

$k_0 := \max\{k_i : 1 \leq i \leq p\}$. Define a filtration $\mathcal{F} := \{F_i : i \geq 0\}$ on $\mathbb{K}[X]$ as follows:

$$\begin{aligned} F_0 &:= \mathbb{K}, \\ F_1 &:= \mathbb{K}\langle 1, x_1, \ldots, x_p, y_1, \ldots, y_q\rangle \ + \ \mathbb{K}\langle y^\beta : |\beta| \leq d_0\rangle \ + \\ &\quad + \mathbb{K}\langle x_i y^\beta : |\beta| \leq d_{i,1}, 1 \leq i \leq p\rangle, \\ F_k &:= \begin{cases} \sum_{j=1}^{k-1} F_j F_{k-j} + \mathbb{K}\langle (x_i)^k y^\beta : |\beta| \leq d_{i,k}, 1 \leq i \leq p\rangle & \text{if } 1 < k \leq k_0, \\ \sum_{j=1}^{k-1} F_j F_{k-j} \quad \text{if } k > k_i \ \forall i. \end{cases} \end{aligned}$$

Let $g := \prod_{i=1}^m g_{i,k_i}$ and $U := \{a \in Y : g(a) \neq 0\}$. Then U is a non-empty Zariski open subset of Y. Let $\xi := (\xi_1, \ldots, \xi_q) : \mathbb{K}^q \to \mathbb{K}^q$ be an arbitrary linear change of coordinates of $\mathbb{K}^q$. It suffices to show that $\psi_{\mathcal{F}}$ preserves the components of $\xi \circ f$ at ∞ over $\xi(a)$ for $a := (a_1, \ldots, a_q) \in U$. For each $a \in Y$, let $H_j(a) := \{x \in X : (\xi_j \circ f)(x) = \xi_j(a)\}$ and let $\mathfrak{q}_j(a)$ be the ideal of $H_j(a)$, i.e. the ideal of $\mathbb{K}[X]$ generated by $\xi_j(y) - \xi_j(a)$. By lemma 1.3.1, $\psi_{\mathcal{F}}$ preserves the components of $\xi \circ f$ at infinity over $\xi(a)$ iff $\sqrt{\mathcal{I}(a)} = \mathbb{K}[X]^{\mathcal{F}}_+$, where $\mathcal{I}(a)$ is the ideal of $\mathbb{K}[X]^{\mathcal{F}}$ generated by $\mathfrak{q}_1^{\mathcal{F}}(a), \ldots, \mathfrak{q}_n^{\mathcal{F}}(a)$ and $(1)_1$. Note the following:

(a) Since ξ is a linear change of coordinate, so is ξ^{-1}. Therefore, for all $d \geq 0$, the $\mathbb{K}$-span of $\{y^\beta : |\beta| \leq d\}$ in $\mathbb{K}[Y]$ is equal to the $\mathbb{K}$-span of $\{(\xi^{-1}(y))^\beta : \deg_y((\xi^{-1}(y))^\beta) \leq d\}$.

(b) If we replace f by $\xi \circ f$, and hence y by $\xi(y)$, then $g_{i,j}(y)$ changes to $g^\xi_{i,j}(y) := g_{i,j}(\xi^{-1}(y)) = \sum_\alpha c_{i,j,\alpha}(\xi^{-1}(y))^\alpha$. But replacing $\sum_\alpha c_{i,j,\alpha} y^\alpha$ by $\sum_\alpha c_{i,j,\alpha}(\xi^{-1}(y))^\alpha$ does *not* change its degree $d_{i,j}$ in y.

(c) Let $g^\xi := \prod_{i=1}^m g^\xi_{i,k_i}$. Then $g^\xi(\xi(a)) \neq 0$ iff $g(a) \neq 0$.

In view of the latter observations and the construction of $\mathcal{F}$ it follows that $\mathcal{F}$ does not change if we replace f by $\xi \circ f$. Moreover, the following two claims are equivalent due to properties (a), (b) and (c) of the preceding paragraph.

(1) $\psi_{\mathcal{F}}$ preserves the components of f at ∞ over $a \in U$, and

(2) $\psi_{\mathcal{F}}$ preserves the components of $\xi \circ f$ at ∞ over $a \in \xi^{-1}(U)$.

Therefore it suffices to prove (1) and we may without loss of generality assume ξ to be the identity. Note that $\mathbb{K}[X]^{\mathcal{F}}_+$ is generated as a $\mathbb{K}$-algebra by elements $(1)_1$, $(x_1)_1, \ldots, (x_p)_1$, $(y_1)_1, \ldots, (y_q)_1$, the $(y^\beta)_1$'s that appear in the definition of F_1, and all those $((x_i)^k y^\beta)_k$ that we inserted in the definition of all F_k's. Therefore $\sqrt{\mathcal{I}(a)} = \mathbb{K}[X]_+^{\mathcal{F}}$ iff some power of each of these generators lies in $\mathcal{I}(a)$.

Lemma 1.3.7.1. *Let a be an arbitrary point in Y.*

1. *Let $\beta \in (\mathbb{Z}_+)^q$ be such that $y^\beta \in F_1$. then*
   (a) *$(y-a)^\beta$ also lies in F_1, and*
   (b) *$((y-a)^\beta)_1 \in \mathcal{I}(a)$.*
2. *Let $1 \le i \le p$. Pick k with $1 \le k \le k_i$ and $\beta \in (\mathbb{Z}_+)^q$ such that $x_i^k y^\beta \in F_k$. Then*
   (a) *$x_i^k (y-a)^\beta$ lies in F_k, and*
   (b) *if in addition $\beta \neq 0$, then $(x_i^k (y-a)^\beta)_k \in \mathcal{I}(a)$.*

Proof. 1. Pick $\beta \in (\mathbb{Z}_+)^q$ such that $y^\beta \in F_1$. Expanding $(y-a)^\beta$ in powers of $y_1, \dots, y_q$, we see that $(y-a)^\beta = \sum_{|\gamma| \le |\beta|} c_\gamma y^\gamma$ for some $c_\gamma \in \mathbb{K}$ and $\gamma \in (\mathbb{Z}_+)^q$. By construction, F_1 contains each of the y^γ appearing in the preceding expression. It follows that F_1 also contains $(y-a)^\beta$, which proves assertion 1a. As for 1b, note that if $\beta = 0$, then $((y-a)^\beta)_1 = (1)_1 \in \mathcal{I}(a)$. Otherwise, there exists j, $1 \le j \le q$, such that the j-th coordinate of β is positive. Then $(y-a)^\beta \in \mathfrak{q}_j(a)$, and hence $((y-a)^\beta)_1 \in \mathfrak{q}_j^{\mathcal{F}}(a) \subseteq \mathcal{I}(a)$, which completes the proof of assertion 1.

2. Let $1 \le i \le p$. Pick k, β such that $1 \le k \le k_i$ and $x_i^k y^\beta \in F_k$. As in the proof of assertion 1, expanding $(y-a)^\beta$ in powers of $y_1, \dots, y_q$, we see that $x_i^k (y-a)^\beta = \sum_{|\gamma| \le |\beta|} c_\gamma x_i^k y^\gamma$ for some $c_\gamma \in \mathbb{K}$. By construction, F_k contains $x_i^k y^\gamma$ for each $|\gamma| \le |\beta|$. It follows that F_k also contains $x_i^k (y-a)^\beta$, and this proves assertion 2a. For 2b, note that if $\beta \neq 0$, then there exists j, $1 \le j \le q$, such that the j-th coordinate of β is positive. Then $x_i^k (y-a)^\beta \in \mathfrak{q}_j(a)$, and hence $(x_i^k (y-a)^\beta)_k \in \mathfrak{q}_j^{\mathcal{F}}(a) \subseteq \mathcal{I}(a)$, which completes the proof of the lemma. □

We now return to the proof of theorem 1.3.7. Let a be any point in Y and β be such that $y^\beta \in F_1$. Expanding y^β in powers of $y_1 - a_1, \dots, y_q - a_q$, we see that $y^\beta = \sum_{|\gamma| \le |\beta|} c'_\gamma (y-a)^\gamma$ for some $c'_\gamma \in \mathbb{K}$. By assertion 1a of lemma 1.3.7.1, each $(y-a)^\gamma$ in the preceding expression lies in F_1. Therefore, in $\mathbb{K}[X]^{\mathcal{F}}$ element $(y^\beta)_1 = \sum_{|\gamma|\leq|\beta|} c'_\gamma((y-a)^\gamma)_1$. But assertion 1b of lemma 1.3.7.1 implies that $((y-a)^\gamma)_1 \in \mathcal{I}(a)$ for all $|\gamma| \leq |\beta|$. Therefore $(y^\beta)_1 \in \mathcal{I}(a)$.

Now expand the polynomial of the left hand side of equation (1.5) (as a polynomial in $y := (y_1, \dots, y_q)$) in powers of $y_1 - a_1, \dots, y_q - a_q$. This leads to an equation of the form

$$\sum_{j=0}^{k_i} g_{i,j}(a)x_i^j + \sum_{\substack{j\leq k_i\\ \beta\neq 0}} h_{i,j,\beta}(a)(y-a)^\beta x_i^j = 0 \ , \tag{1.5$'$}$$

where terms $(y-a)^\beta x_i^j$, for $\beta \neq 0$ and $j \leq k_i$, that appear in (1.5′) are such that $y^\beta x_i^j \in F_j$, and due to assertion 2a of lemma 1.3.7.1, $(y-a)^\beta x_i^j$ are in $F_j \subseteq F_{k_i}$. Therefore equation (1.5′) implies the following equality in $\mathbb{K}[X]^{\mathcal{F}}$:

$$g_{i,k_i}(a)((x_i)_1)^{k_i} = -\sum_{j=0}^{k_i-1} g_{i,j}(a)((x_i)_1)^j((1)_1)^{k_i-j} \ - \sum_{\substack{j\leq k_i\\ \beta\neq 0}} h_{i,j,\beta}(a)((y-a)^\beta x_i^j)_{k_i}. \tag{1.5$''$}$$

Since $(1)_1 \in \mathcal{I}(a)$, each of the terms under the first summation of the right hand side of (1.5″) lies in $\mathcal{I}(a)$. Moreover, by assertion 2b of lemma 1.3.7.1, every $((y-a)^\beta x_i^j)_{k_i}$ under the second summation of the right hand side of (1.5″) also belongs to $\mathcal{I}(a)$. It follows that $g_{i,k_i}(a)((x_i)_1)^{k_i} \in \mathcal{I}(a)$, and therefore $(x_i)_1 \in \sqrt{\mathcal{I}(a)}$ if $g_{i,k_i}(a) \neq 0$. Hence for all $a \in U$ and i, $1 \leq i \leq p$, elements $(x_i)_1 \in \sqrt{\mathcal{I}(a)}$.

Let $a \in U$ and $1 \leq i \leq p$. Pick $(x_i)^k y^\beta \in F_k$ with $1 \leq k \leq k_i$. Expanding y^β in powers of $y_1 - a_1, \dots, y_q - a_q$, we see that $(x_i)^k y^\beta = \sum_{|\gamma|\leq|\beta|} c'_\gamma (x_i)^k (y-a)^\gamma$ for some $c'_\gamma \in \mathbb{K}$. By assertion 2a of lemma 1.3.7.1, each $(x_i)^k(y-a)^\gamma$ in the preceding expression lies in F_k. Therefore in $\mathbb{K}[X]^{\mathcal{F}}$ element $((x_i)^k y^\beta)_k = \sum_{|\gamma|\leq|\beta|} c'_\gamma((x_i)^k(y-a)^\gamma)_k$. If $|\gamma| \leq |\beta|$ and $\gamma \neq 0$, assertion 2b of lemma 1.3.7.1 implies that $((x_i)^k(y-a)^\gamma)_k \in \mathcal{I}(a)$ and if $\gamma = 0$, then $((x_i)^k(y-a)^\gamma)_k = ((x_i)^k)_k = ((x_i)_1)^k \in \sqrt{\mathcal{I}(a)}$ due to the conclusion of the preceding paragraph. Therefore for $x_i^k y^\beta \in F_k$, $1 \leq k \leq k_i$, it follows that $((x_i)^k y^\beta)_k \in \sqrt{\mathcal{I}(a)}$.

Consequently $\sqrt{\mathcal{I}(a)} = \mathbb{K}[X]^{\mathcal{F}}_+$ for all $a \in U$, which completes the proof. □

Example 1.3.8. Let $X = Y = \mathbb{C}^2$ and $f := (f_1, f_2) : X \to Y$ be the map defined by $f_1 := x_1^3 + x_1^2 x_2 + x_1 x_2^2 - c x_2$ and $f_2 := x_1^3 + x_1^2 x_2 + x_1 x_2^2 - x_2$ for any complex number $c \neq 0$ or 1. Note that this map is a minor variation of the map considered in remark 1.3.6. Let $\psi : X \hookrightarrow \bar{X} \subseteq \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the completion considered in theorem 1.3.5 and remark 1.3.6, i.e. $\psi(x_1, x_2) := ([1 : x_1 : x_2], [1 : f_1(x_1, x_2)], [1 : f_2(x_1, x_2)])$ for all $(x_1, x_2) \in X$. Below we show that ψ does *not* satisfy the preservation property of theorem 1.3.7.

For each $\lambda := (\lambda_1, \lambda_2) \in \mathbb{C}^2$ let

$$f_\lambda := \lambda_1 f_1 + \lambda_2 f_2 = (\lambda_1 + \lambda_2) x_1^3 + (\lambda_1 + \lambda_2) x_1^2 x_2 + (\lambda_1 + \lambda_2) x_1 x_2^2 - (\lambda_1 c + \lambda_2) x_2 \,.$$

Fix $\lambda^1, \lambda^2 \in \mathbb{C}^2 \setminus (L_1 \cup L_2 \cup L_3)$ where L_1 is the x_1-axis, L_2 is the x_2-axis and L_3 is the line $\{(x_1, x_2) \in \mathbb{C}^2 : x_1 + x_2 = 0\}$. Fix an $a := (a_1, a_2) \in Y$. Let $b_i := \lambda_1^i a_1 + \lambda_2^i a_2$, $i = 1, 2$. Every $H_i(a) = \{x \in X : f_{\lambda^i}(x) = b_i\}$ is as in theorem 1.3.7, $i = 1, 2$. Let $C_i(a)$ be the closure in $\mathbb{P}^2(\mathbb{C})$ of $H_i(a)$ for $i = 1, 2$. Then $P := [0 : 0 : 1] \in C_1(a) \cap C_2(a)$, where as in remark 1.3.6, we choose coordinates $[Z : X_1 : X_2]$ on $\mathbb{P}^2(\mathbb{C})$ and identify X with $\mathbb{P}^2(\mathbb{C}) \setminus V(Z)$. Choose local coordinates $\xi_1 := X_1/X_2$ and $\xi_2 := Z/X_2$ on $\mathbb{P}^2(\mathbb{C})$ near $P := [0 : 0 : 1]$. Equations of $f_{\lambda^i}(x_1, x_2) - b_i$ in (ξ_1, ξ_2) coordinates are:

$$f_{\lambda^i}(x_1, x_2) - b_i = (\lambda_1^i + \lambda_2^i)\xi_1^3 + (\lambda_1^i + \lambda_2^i)\xi_1^2 + (\lambda_1^i + \lambda_2^i)\xi_1 - (\lambda_1^i c + \lambda_2^i)\xi_2^2 - a_i \xi_2^3 \,.$$

Since $\lambda_1^i + \lambda_2^i \neq 0$, it follows (similarly to the implication (1.3) $\Rightarrow$ (1.4) of remark 1.3.6) that for each i, $C_i(a)$ is smooth at P and has a parametrization at P of the form

$$\gamma_{i,a}(t) := [t : \frac{\lambda_1^i c + \lambda_2^i}{\lambda_1^i + \lambda_2^i} t^2 + o(t^3) : 1] \,.$$

In (x_1, x_2) coordinates the parametrizations are: $(x_1(t), x_2(t)) = (\frac{\lambda_1^i c + \lambda_2^i}{\lambda_1^i + \lambda_2^i} t + o(t^2), 1/t)$ for $t \in \mathbb{C}^*$. A straightforward calculation shows that for $t \in \mathbb{C}^*$

$$\begin{aligned} f_1(\gamma_{i,a}(t)) &= \frac{\lambda_2^i (1 - c)}{\lambda_1^i + \lambda_2^i} \frac{1}{t} + o(t^2) \quad \text{and} \\ f_2(\gamma_{i,a}(t)) &= \frac{\lambda_1^i (c - 1)}{\lambda_1^i + \lambda_2^i} \frac{1}{t} + o(t^2) \,, \end{aligned} \tag{1.6}$$

$i = 1, 2$. By our choice of $c \neq 0, 1$ and λ^i's off $L_1 \cup L_2 \cup L_3$ it follows that the coefficients at $\frac{1}{t}$ in the expressions in (1.6) are non-zero. It follows that $\lim_{t\to 0} |f_j(\gamma_{i,a}(t))| = \infty$ for each i, j, and therefore

$$\begin{aligned}\lim_{t\to 0} \psi(\gamma_{i,a}(t)) &= \lim_{t\to 0}([1 : \gamma_{i,a}(t)], [1 : f_1(\gamma_{i,a}(t))], [1 : f_2(\gamma_{i,a}(t))]) \\ &= ([0 : 0 : 1], [0 : 1], [0 : 1]) \ .\end{aligned}$$

Hence $([0 : 0 : 1], [0 : 1], [0 : 1])$ is in the closure of $H_i(a)$ in $\bar{X}$ for $i = 1, 2$ and every $a \in Y$. It follows that ψ does not preserve $(f_{\lambda^1}, f_{\lambda^2})$ at ∞ over *any* point in Y, as claimed.

Finally, we follow the proof of theorem 1.3.7 to find a completion which preserves map f at ∞. The algebraic equations satisfied by x_1 and x_2 over $\mathbb{C}[f_1, f_2]$ are:

$$\begin{gathered} x_1^3 + \frac{1}{1-c}(f_1 - f_2)x_1^2 + \frac{1}{(1-c)^2}(f_1 - f_2)^2 x_1 - \frac{1}{1-c}(f_1 - cf_2) = 0, \ \text{and} \\ x_2 - \frac{1}{1-c}(f_1 - f_2) = 0. \end{gathered}$$

In the notations of the proof of theorem 1.3.7, $d_{1,2} = d_{1,0} = 1$, $d_{1,1} = 2$ and $d_{2,0} = 1$. Let $\mathcal{F} := \{F_d : d \geq 0\}$ be the filtration defined by:

$$F_0 := \mathbb{K},\ F_1 := \mathbb{K}\langle 1, x_1, x_2, f_1, f_2, x_1f_1, x_1f_2, x_1f_1^2, x_1f_1f_2, x_1f_2^2\rangle,\ F_d := (F_1)^d \text{ for } d \geq 2\ .$$

Construction of the proof of 1.3.7 yields that $X^{\mathcal{F}}$ preserves map f at ∞. Indeed, $\psi_{\mathcal{F}}(x) = [1 : x_1 : x_2 : f_1(x) : f_2(x) : x_1f_1(x) : x_1f_2(x) : x_1f_1^2(x) : x_1f_1(x)f_2(x) : x_1f_2^2(x)] \in \mathbb{P}^9(\mathbb{C})$ for $x \in \mathbb{C}^2$ and therefore applying (1.6) it follows that $\lim_{t\to 0} \psi_{\mathcal{F}}(\gamma_{i,a}(t)) =$

$$\begin{aligned} &\lim_{t\to 0}[1 : \left(\frac{\lambda_1^i c + \lambda_2^i}{\lambda_1^i + \lambda_2^i} t + o(t^2)\right) : \frac{1}{t} : \left(\frac{\lambda_2^i(1-c)}{\lambda_1^i + \lambda_2^i}\frac{1}{t} + o(t^2)\right) : \left(\frac{\lambda_1^i(c-1)}{\lambda_1^i + \lambda_2^i}\frac{1}{t} + o(t^2)\right) : o(1) : \\ &\quad o(1) : \left(\frac{(1-c)^2(\lambda_2^i)^2(\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3}\frac{1}{t} + o(t^2)\right) : \left(-\frac{(1-c)^2\lambda_1^i\lambda_2^i(\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3}\frac{1}{t} + o(t^2)\right) : \\ &\quad \left(\frac{(1-c)^2(\lambda_1^i)^2(\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3}\frac{1}{t} + o(t^2)\right)] \\ &= [0 : 0 : 1 : \frac{\lambda_2^i(1-c)}{\lambda_1^i + \lambda_2^i} : \frac{\lambda_1^i(c-1)}{\lambda_1^i + \lambda_2^i} : 0 : 0 : \frac{(1-c)^2(\lambda_2^i)^2(\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3} : \\ &\quad -\frac{(1-c)^2\lambda_1^i\lambda_2^i(\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3} : \frac{(1-c)^2(\lambda_1^i)^2(\lambda_1^i c + \lambda_2^i)}{(\lambda_1^i + \lambda_2^i)^3}] \ . \end{aligned}$$

If λ^1 and λ^2 are linearly independent, then it follows that limits $\lim_{t \to 0} \psi_{\mathcal{F}}(\gamma_{i,a}(t))$ are different for $i = 1, 2$, and therefore, completion $\psi_{\mathcal{F}}$ preserves $\{f_{\lambda^1}, f_{\lambda^2}\}$ at ∞ over *every* $a \in Y = \mathbb{C}^2$.

Chapter 2

Intersection preserving filtrations with nilpotent-free graded rings

We show that in the main theorem of chapter 1 it suffices to take a restricted class of filtrations, namely those with nilpotent-free graded rings. A filtration $\mathcal{F} := \{F_d : d \in \mathbb{Z}\}$ on the coordinate ring A of an affine variety is, in general, uniquely determined by the corresponding *degree like function* $\delta : A \setminus \{0\} \to \mathbb{Z}$ via $F_d = \{f : \delta(f) \leq d\}$. The second main theorem of this chapter (which appears first!) states that if the graded ring $\operatorname{gr} A^{\mathcal{F}} := \bigoplus F_d/F_{d-1}$ has no nilpotents, then the associated degree like function, which we then call a *subdegree*, is the maximum of a finite collection of degree like functions that send products into sums. We also show that completions determined by subdegrees have various other properties resembling toric completions.

2.0 Background

In section 2.0.1 we briefly introduce projective toric varieties, which will form a large part of the examples in this chapter. Here we mostly follow the exposition in [11]. We describe in section 2.0.2 the notion of Cartier and Weil divisors on arbitrary varieties following [10].

2.0.1 Lattice points and Toric Varieties

The affine variety $(\mathbb{C}^*)^n$ is a group under component-wise multiplication. It is actually an *algebraic group*, meaning the group multiplication map $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ and the inverse map $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ are regular maps of varieties. An n dimensional *torus* is an affine variety T isomorphic to $(\mathbb{C}^*)^n$, where T inherits a group structure from the isomorphism. A complex algebraic variety X is a *toric variety* if it contains a torus T as a Zariski open subset and the action of T on itself extends continuously to an action of T on X.

Example 2.0.1. $(\mathbb{C}^*)^n$ is an affine toric variety, so is $\mathbb{C}^n$ for all n. The weighted projective spaces are examples of projective toric varieties.

In this text, a *lattice* is a free abelian group of finite rank and a *lattice homomorphism* is a homomorphism of abelian groups between two lattices. An *affine transformation* between lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ is a map $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ which is of the form $\phi(\cdot) = \psi(\cdot) + \alpha$ for a lattice homomorphism ψ and a fixed $\alpha \in \mathcal{L}_2$. Let $\mathcal{A} := \{\alpha_0, \dots, \alpha_k\}$ be a finite subset of $\mathbb{Z}^L$ for some $L \geq 0$. The *affine hull* of $\mathcal{A}$ is $\operatorname{aff}(\mathcal{A}) := \{\sum_{i=0}^k a_i\alpha_i : a_i \in \mathbb{Z}\,,\ \sum_{i=0}^k a_i = 1\}$. There is a unique sublattice $\mathcal{L}$ of $\mathbb{Z}^L$ such that $\operatorname{aff}(\mathcal{A}) = \mathcal{L} + \alpha$ for any $\alpha \in \operatorname{aff}(\mathcal{A})$. *Dimension* of $\operatorname{aff}(\mathcal{A})$ is the rank of $\mathcal{L}$ as a free abelian group. Let $\phi_{\mathcal{A}} : (\mathbb{C}^*)^L \to \mathbb{P}^k(\mathbb{C})$ be the map defined by $\phi_{\mathcal{A}}(x) := [x^{\alpha_0} : \cdots : x^{\alpha_k}]$ and let $X_{\mathcal{A}}$ be the closure of $\phi_{\mathcal{A}}((\mathbb{C}^*)^L)$ in $\mathbb{P}^k(\mathbb{C})$.

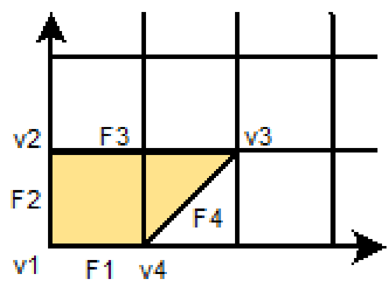
Theorem 2.0.2.

1. *$X_{\mathcal{A}}$ is a toric variety. The torus T of $X_{\mathcal{A}}$ is precisely $\phi_{\mathcal{A}}((\mathbb{C}^*)^L)$. Dimension of $X_{\mathcal{A}}$ is the same as the dimension of* $\operatorname{aff}(\mathcal{A})$*, say n. In particular, T is isomorphic to $(\mathbb{C}^*)^n$. The action of T on $X_{\mathcal{A}}$ is given by coordinatewise multiplication, i.e.* $[t_0 : \cdots : t_k] \cdot [x_0 : \cdots : x_k] := [t_0x_0 : \cdots : t_kx_k]$.
2. *For any $K \geq 0$, if $\mathcal{B}$ is a finite subset of $\mathbb{Z}^K$ and $\phi : \mathbb{Z}^K \to \mathbb{Z}^L$ is an affine transformation such that $\phi|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ is an isomorphism of sets, then $X_{\mathcal{B}}$ is*

isomorphic to $X_{\mathcal{A}}$ *as toric varieties.*

3. *Let* $\mathcal{P}$ *be the convex hull of* $\mathcal{A}$ *in* $\mathbb{R}^L$*. There is a one-to-one correspondence between the faces of* $\mathcal{P}$ *and torus orbits of* $X_{\mathcal{A}}$*. The orbit corresponding to a face* $\mathcal{Q}$ *of* $\mathcal{P}$ *is* $O_{\mathcal{Q}} := \{[z_0 : \ldots : z_k] \in X_{\mathcal{A}} : z_i = 0 \text{ iff } \alpha_i \notin \mathcal{Q}\}$*. The closure of* $O_{\mathcal{Q}}$ *is* $V_{\mathcal{Q}} := \{[z_0 : \ldots : z_k] \in X_{\mathcal{A}} : z_i = 0 \text{ if } \alpha_i \notin \mathcal{Q}\}$*.* $V_{\mathcal{Q}}$ *is a subvariety of* $X_{\mathcal{Q}}$ *which is isomorphic to* $X_{\mathcal{A}\cap\mathcal{Q}}$ *and its dimension is the same as the Euclidean dimension of* $\mathcal{Q}$*. The correspondence* $\mathcal{Q} \longleftrightarrow V_{\mathcal{Q}}$ *is* inclusion preserving*, i.e.* $\mathcal{Q} \subseteq \mathcal{R}$ *iff* $V_{\mathcal{Q}} \subseteq V_{\mathcal{R}}$*. It follows that* $X_{\mathcal{A}} \setminus T = \bigcup\{V_{\mathcal{Q}} : \mathcal{Q}$ *is a* facet *(i.e. codimension one face) of* $\mathcal{P}\}$*.*
4. *For each* $m \geq 1$*, define* $\mathcal{A}_m := m\mathcal{P} \cap \mathbb{Z}^N$*. Then the toric varieties* $X_{\mathcal{A}_m}$ *are isomorphic for* $m \geq n-1$*. Let* $X_{\mathcal{P}}$ *denote any representative of the isomorphism class of these varieties. Then* $X_{\mathcal{P}}$ *is the* normalization *of* $X_{\mathcal{A}}$*; in particular, it is normal.*

Example 2.0.3. Let $\mathcal{A} := \{(0,0),(1,0),(0,1),(2,1)\} \subseteq \mathbb{Z}^2$. The convex hull $\mathcal{P}$ of $\mathcal{A}$ is drawn in the picture below, with its faces and vertices marked.



Embedding $\phi_{\mathcal{A}} : (\mathbb{C}^*)^2 \ni (x,y) \mapsto [1 : x : y : x^2y] \in \mathbb{P}^3(\mathbb{C})$. Variety $\phi_{\mathcal{A}}((\mathbb{C}^*)^2)$ is a group under coordinatewise multiplication, and map $\phi_{\mathcal{A}} : (\mathbb{C}^*)^2 \to \phi_{\mathcal{A}}((\mathbb{C}^*)^2)$ is a group *isomorphism* with the inverse being $[W : X : Y : Z] \mapsto (\frac{X}{W}, \frac{Y}{W})$. Thus we see that $X_{\mathcal{A}}$ is indeed a toric variety with its torus $T := \phi_{\mathcal{A}}((\mathbb{C}^*)^2) \cong (\mathbb{C}^*)^2$.

In homogeneous coordinates $[W : X : Y : Z]$ on $\mathbb{P}^3(\mathbb{C})$, subvariety $X_{\mathcal{A}}$ coincides with $V(W^2Z - X^2Y)$, and a straightforward calculation shows that $T = X_{\mathcal{A}} \setminus V(WXYZ)$. Therefore $X_{\mathcal{A}} \setminus T = V(W^2Z - X^2Y, WXYZ) = V(Y,Z) \cup V(Z,X) \cup V(X,W) \cup V(W,Y)$.

Note that each of the four components are 1 dimensional (in fact isomorphic to $\mathbb{P}^1(\mathbb{C})$) and they are preserved under the action of T. In the notation of assertion 3 of theorem 2.0.2 these components are precisely $V_{F_1}, \dots, V_{F_4}$ (the correspondence being in the same order as listed). The zero dimensional orbits, or fixed points of the action of T, are $V_{v_1} = V(Y, Z, X)$, $V_{v_2} = V(Z, X, W)$, $V_{v_3} = V(X, W, Y)$ and $V_{v_4} = V(W, Y, Z)$.

Note that $X_{\mathcal{A}}$ is *not* normal. Indeed, let $\xi_1 := W/Z$, $\xi_2 := X/Z$ and $\xi_3 := Y/Z$ be coordinates of $U_Z := \mathbb{P}^3(\mathbb{C}) \setminus V(Z)$. Then $X_{\mathcal{A}} \cap U_Z = V(\xi_1^2 - \xi_2^2\xi_3)$. Therefore the coordinate ring of $X_{\mathcal{A}} \cap U_Z$ is $A := \mathbb{C}[\xi_1, \xi_2, \xi_3]/\langle \xi_1^2 - \xi_2^2\xi_3 \rangle \cong \mathbb{C}[\xi_1, \xi_2, \xi_1^2/\xi_2^2]$. But then $(\xi_1/\xi_2)^2 \in A$, even though $\xi_1/\xi_2 \notin A$. It follows that A is not integrally closed. In example 2.0.6 we will calculate $X_{\mathcal{P}}$ and show that it is the normalization of $X_{\mathcal{A}}$.

2.0.2 Divisors

Let A be a commutative ring. A *chain* of prime ideals of a ring A is a finite increasing sequence $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$, where each $\mathfrak{p}_i$ is a prime ideal of A; the *length* of the chain is n. The *Krull dimension* of A, which we will refer to simply as *dimension* of A, is the supremum of the lengths of all chains of prime ideals in A. Any field has dimension 0; the ring of integers has dimension 1. The dimension of the coordinate ring of any affine algebraic variety X over an algebraically closed field $\mathbb{K}$ coincides with the dimension of X (which is the transcendence degree over $\mathbb{K}$ of the field of $\mathbb{K}(X)$ of rational functions on X).

A *chain* of submodules of an A-module M of *length* n is an increasing sequence $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n$, where each M_i is an A-submodule of M. The supremum of the lengths of all chains of submodules of M is called the *length* of M. Let R be a one dimensional Noetherian domain and K be the quotient field of R. For any nonzero element $r \in R$, $R/\langle r \rangle$ has finite length as an R-module. Denote this length by $\operatorname{ord}_R(r)$.

Proposition 2.0.4 ([10, Appendix A3])**.** *Map* $\operatorname{ord}_R : r \to \operatorname{ord}_R(r)$ *extends to a group homomorphism from the multiplicative group* $K^* := K \setminus \{0\}$ *to* $\mathbb{Z}$*. Let* ord_R *denote the* *latter extension as well. Then* $\operatorname{ord}_R(a/b) := \operatorname{ord}_R(a) - \operatorname{ord}_R(b)$ *for every non-zero* $a, b \in R$.

Let V be a codimension one subvariety of a variety Z. The local ring $R := \mathcal{O}_{V,Z}$ of Z at V is a one-dimensional local domain. For a nonzero $r \in \mathbb{K}(Z)$, define $\operatorname{ord}_V(r) := \operatorname{ord}_R(r)$, where ord_R is defined as above. A *Weil divisor* on Z is a finite formal sum $\sum n_i[V_i]$, where V_i are the codimension one subvarieties of Z, and n_i are integers. Its *support* is the union of all V_i such that $n_i \neq 0$, and it is *effective* if $n_i \geq 0$ for all i. A *Cartier divisor* D on Z is defined by data (U_α, f_α), where $\{U_\alpha\}_\alpha$ is an open covering of Z and f_α are non-zero elements in $\mathbb{K}(Z)$, subject to the condition that f_α/f_β is a unit (i.e. a nowhere vanishing regular function) on $U_\alpha \cap U_\beta$. For each α, f_α is called the *local equation* of D on U_α. If in addition each f_α is a regular function on U_α, then D is an *effective* Cartier divisor. If V is a subvariety of Z of codimension one, let $\operatorname{ord}_V(D) := \operatorname{ord}_V(f_\alpha)$ for any α such that $U_\alpha \cap V \neq \emptyset$. The *Weil divisor associated* to D is $[D] := \sum \operatorname{ord}_V(D)[V]$, where the sum is over all codimension one subvarieties V of Z; this sum is finite, since there are only finitely many V with $\operatorname{ord}_V(D) \neq 0$. If D is an effective Cartier divisor, then $[D]$ is an effective Weil divisor. Let $\operatorname{Supp}(D) := \bigcup\{V : \operatorname{ord}_V(D) \neq 0\}$ denote the *support* of divisor D.

Two Cartier divisors D and D' are linearly equivalent if there exists a nonzero rational function r on Z such that for all $z \in Z$, the quotient of the local equations of D and of D' at z is r. The *Picard group* $\operatorname{Pic}(Z)$ of Z is the group of Cartier divisors on Z modulo linear equivalence. On the other hand, every nonzero rational function r on Z defines a Weil divisor $[\operatorname{div}(r)] := \sum \operatorname{ord}_V(r)[V]$ called the *principal divisor* corresponding to r. The quotient of the group of Weil divisors of Z by the subgroup of principal divisors is called the *divisor class group* of Z, and is denoted by $\operatorname{Cl} Z$. The correspondence $D \to [D]$ induces a homomorphism $\operatorname{Pic}(Z) \to \operatorname{Cl}(Z)$ which is an embedding when Z is normal.

The theory of divisors on normal toric varieties is particularly tractable. Let $\mathcal{A} \subseteq \mathbb{Z}^L$ be as in section 2.0.1, with $\mathcal{P} :=$ the convex hull of $\mathcal{A}$ in $\mathbb{R}^L$. Assume that $\operatorname{aff}(\mathcal{A}) = \mathbb{Z}^L$ (by assertion 2 of theorem 2.0.2, this does not change the class of corresponding toric varieties). Then each facet $\mathcal{Q}$ of $\mathcal{P}$ admits the 'smallest inner normal' $\eta_{\mathcal{Q}}$: in other words $\eta_{\mathcal{Q}}$ is the unique generator of the semigroup $\{\eta \in \mathbb{Z}^L : \langle \eta, \alpha \rangle \leq \langle \eta, \beta \rangle$ for all $\alpha \in \mathcal{Q}$ and $\beta \in \mathcal{P}\}$, where for all $\gamma \in \mathbb{Z}^L$, $\langle \eta, \gamma \rangle := \sum_{i=1}^{L} \eta_i \gamma_i$ is the usual inner product. Note that by assertion 1 of theorem 2.0.2, $\dim(X_{\mathcal{P}}) = L$, and we can (and will) identify $(\mathbb{C}^*)^L$ with the torus T of $X_{\mathcal{A}}$ via the isomorphism induced by $\phi_{\mathcal{P}}$.

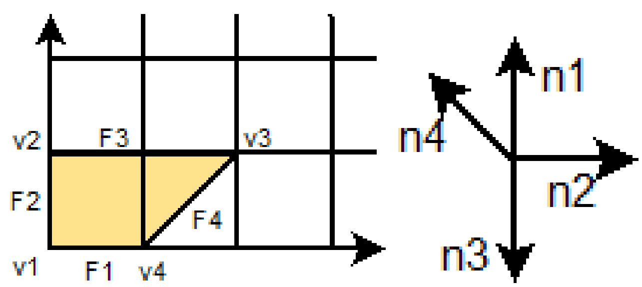
Theorem 2.0.5 ([9, Chapter 3])**.** *For each facet $\mathcal{Q}$ of $\mathcal{P}$, let $V_{\mathcal{Q}}$ be the torus invariant subvariety corresponding to $\mathcal{Q}$ (as in assertion 3 of theorm 2.0.2).*

1. *The principal divisor of x^α on $X_{\mathcal{A}}$ is $[x^\alpha] = \sum_{\mathcal{Q}} \langle \eta_{\mathcal{Q}}, \alpha \rangle [V_{\mathcal{Q}}]$, where the sum is over the facets $\mathcal{Q}$ of $\mathcal{P}$.*
2. *$\{[V_{\mathcal{Q}}] : \mathcal{Q}$ is a facet of $\mathcal{P}\}$ generate $\mathrm{Cl}(X_{\mathcal{P}})$ and there is a commutative diagram with exact rows:*

$$0 \to \mathbb{Z}^L \to \bigoplus \mathbb{Z}[V_{\mathcal{Q}}] \to \mathrm{Cl}(X_{\mathcal{P}}) \to 0,$$

where the map $\mathbb{Z}^L \to \bigoplus \mathbb{Z}[V_{\mathcal{Q}}]$ is defined by $\alpha \mapsto \sum_{\mathcal{Q}} \langle \eta_{\mathcal{Q}}, \alpha \rangle [V_{\mathcal{Q}}]$.

Example 2.0.6. Let $\mathcal{A}$ and $\mathcal{P}$ be as in example 2.0.3. Below we reproduce the picture of $\mathcal{P}$ along with the inner normals of its facets.



According to assertion 4 of theorem 2.0.2, variety $X_{\mathcal{P}} \cong X_{\mathcal{P} \cap \mathbb{Z}^2}$. Since $\mathcal{P} \cap \mathbb{Z}^2 = \{(0,0), (1,0), (0,1), (1,1), (2,1)\}$, map $\phi_{\mathcal{P}}$ sends $(x, y) \in (\mathbb{C}^*)^2$ to $[1 : x : y : xy : x^2 y] \in \mathbb{P}^4(\mathbb{C})$. Let the homogeneous coordinates on $\mathbb{P}^4(\mathbb{C})$ be $[W : X : Y : T : Z]$. A straightforward calculation shows that the ideal of $X_{\mathcal{P}} \hookrightarrow \mathbb{P}^4(\mathbb{C})$ is generated by polynomials $(TW - XY)$, $(ZW - XT)$ and $(T^2 - YZ)$. Let $(x, y, z, t) := (X/W, Y/W, Z/W, T/W)$ be the affine coordinates on $U_W := \mathbb{P}^4(\mathbb{C}) \setminus V(W)$. Then in U_W, variety $X_{\mathcal{P}} \cap U_W =$

$V(t-xy, z-xt, t^2-yz) = V(t-xy, z-xt) \cong \mathbb{C}^2$. Similary one can show that $X_{\mathcal{P}} \setminus V(G) \cong \mathbb{C}^2$ for the polynomials G being equal to any of the remaining homogeneous coordinates. It follows that $X_{\mathcal{P}}$ is non-singular; in particular it is normal. The equations of the subvarieties V_i corresponding to faces F_i are: $V_1 = V(Y, T, Z)$, $V_2 = V(T, Z, X)$, $V_3 = V(X, W, T^2 - YZ)$ and $V_4 = V(W, Y, T)$. The restriction to $X_{\mathcal{P}}$ of projection $\pi : [W : X : Y : T : Z] \mapsto [W : X : Y : Z]$ gives a well defined map from $X_{\mathcal{P}}$ to $X_{\mathcal{A}}$. Then π maps the torus of $X_{\mathcal{P}}$ isomorphically onto the torus of $X_{\mathcal{A}}$, and maps V_i onto $V_i(X_{\mathcal{A}})$ where $V_i(X_{\mathcal{A}})$ are the codimension one subvarieties of $X_{\mathcal{A}}$ corresponding to F_i (the subvarieties we denoted by V_i in example 2.0.3). In particular it is a finite birational map, and therefore it is the normalization map (by the universal property of normalization).

A straightforward calculation shows that map $\phi_{\mathcal{P}} : \mathbb{C}^2 \to \mathbb{P}^4(\mathbb{C})$ defined by $\phi_{\mathcal{P}} : (x, y) \mapsto [1 : x : y : xy : x^2y]$ embeds $\mathbb{C}^2$ isomorphically onto $X_{\mathcal{P}} \setminus V(W)$. Then V_1 and V_2 are the closures of the x and, respectively, of the y-axis of $\mathbb{C}^2$. It follows that for each $\alpha := (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, $\mathrm{ord}_{V_1}(x^{\alpha_1}y^{\alpha_2}) = \alpha_2$ and $\mathrm{ord}_{V_2}(x^{\alpha_1}y^{\alpha_2}) = \alpha_1$. To calculate $\mathrm{ord}_{V_3}(\cdot)$ and $\mathrm{ord}_{V_4}(\cdot)$, it suffices to examine the affine chart $U_Z := \mathbb{P}^4(\mathbb{C}) \setminus V(Z)$ with coordinates $(\xi_1, \xi_2, \xi_3, \xi_4) := (T/Z, X/Z, Y/Z, W/Z)$. Then in U_Z variety $X_{\mathcal{P}} \cap U_Z$ coincides with $V(\xi_1\xi_4 - \xi_2\xi_3\,, \xi_4 - \xi_2\xi_1\,, \xi_1^2 - \xi_3)$. It follows that $X_{\mathcal{P}}$ is a graph over the (ξ_1, ξ_2)-plane of the map $f(\xi_1, \xi_2) := (\xi_1^2, \xi_1\xi_2)$ and projection to (ξ_1, ξ_2)-plane maps $X_{\mathcal{P}} \cap U_Z$ isomorphically onto (ξ_1, ξ_2)-plane. Under this isomorphism $V_3 \cap U_Z$ and $V_4 \cap U_Z$ are mapped to the ξ_1 and, respectively, to the ξ_2-axis. Note that $x = X/W = \xi_2/\xi_4 = \xi_2/(\xi_1\xi_2) = 1/\xi_1$, and $y = Y/W = \xi_3/\xi_4 = \xi_1^2/(\xi_1\xi_2) = \xi_1/\xi_2$. Hence $x^{\alpha_1}y^{\alpha_2} = \xi_1^{\alpha_2-\alpha_1}\xi_2^{-\alpha_2}$ and therefore $\mathrm{ord}_{V_3}(x^{\alpha_1}y^{\alpha_2}) = -\alpha_2$ and $\mathrm{ord}_{V_4}(x^{\alpha_1}y^{\alpha_2}) = \alpha_2 - \alpha_1$. Finally, observe that the smallest inner normals associated to the F_i are: $\eta_1 = (0, 1)$, $\eta_2 = (1, 0)$, $\eta_3 = (0, -1)$ and $\eta_4 = (-1, 1)$ and, indeed, $\mathrm{ord}_{V_i}(x^{\alpha_1}y^{\alpha_2}) = \langle \eta_i, \alpha \rangle$ for all $\alpha \in \mathbb{Z}^2$.

2.1 Semidegree and Subdegree

Throughout the remainder of this chapter let A denote a finitely generated domain over any field $\mathbb{K}$.

Definition. A *degree like function* on A is a map $\delta : A \setminus \{0\} \to \mathbb{Z} \cup \{-\infty\}$ such that:

1. $\delta(\mathbb{K}) = 0$.
2. $\delta(f + g) \leq \max\{\delta(f), \delta(g)\}$ for all $f, g \in A$, with $<$ in the preceding equation implying $\delta(f) = \delta(g)$.
3. $\delta(fg) \leq \delta(f) + \delta(g)$ for all $f, g \in A$.

Remark. Even though we allow degree like functions to have values $-\infty$, it will follow from theorems 2.2.11 and 2.2.16 that we do not need to leave the realm of integer valued degree like functions by either of the operations of *normalizing* degree like functions or *taking associated semidegrees of a subdegree* (we introduce both notions below). Intermediately we allow for a theoretical possibility of ending up with a degree like function that takes the value $-\infty$ on some nonzero $f \in A$.

There is a one-to-one correspondence between degree like functions and filtrations:

Filtrations	$\longleftrightarrow$	*Degree like functions*
$\mathcal{F} = \{F_d\}_{d\in\mathbb{Z}}$	$\longrightarrow$	$\delta_{\mathcal{F}} : f \in A \mapsto \inf\{d : f \in F_d\}$
$\mathcal{F}_\delta := \{F_d := \{f \in A : \delta(f) \leq d\}\}_{d\in\mathbb{Z}}$	$\longleftarrow$	δ

In the remainder of our thesis we identify degree like functions with the corresponding filtrations. In particular, we refer to a degree like function δ as *complete* (resp. *finitely generated*) iff the corresponding filtration $\mathcal{F}_\delta$ is complete (resp. finitely generated). Moreover, A^δ and $\operatorname{gr} A^\delta$ will be shorthand notations for the rings $A^{\mathcal{F}_\delta}$ and, respectively, $\operatorname{gr} A^{\mathcal{F}_\delta}$ and ψ_δ will denote the natural embedding $\operatorname{Spec} A \hookrightarrow X^\delta := \operatorname{Proj} A^\delta$, while for the sake of convenience we would freely refer to X^δ as the "completion ψ_δ" (of $\operatorname{Spec} A$).

The protagonists of this chapter are two classes of degree like functions which satisfy stronger versions of the multiplicative property (i.e. property 3 above).

Definition.

- A degree like function δ on A is a *semidegree* iff $\delta(fg) = \delta(f) + \delta(g)$ for all $f, g \in A \setminus \{0\}$.
- We say that δ is a *subdegree* if there are semidegrees $\delta_1, \ldots, \delta_N$ such that

$$\delta(f) = \max_{1 \le i \le N} \delta_i(f) \quad \text{for all } f \in A \setminus \{0\}. \tag{2.1}$$

Given a subdegree δ as in (2.1), we may assume by getting rid of some δ_i's, if need be, that every δ_i that appears in (2.1) is *not redundant* in the sense that for every i, there is an $f \in A$ such that $\delta_i(f) > \delta_j(f)$ for all $j \neq i$ (indeed, if there is an i such that for all $f \in A$, $\delta_i(f) \le \delta_j(f)$ for some $j \neq i$, then $\delta(f) = \max_{j \neq i} \delta_j(f)$ for all $f \in A$). In the latter case we will say that (2.1) is a *minimal presentation* of δ.

Example 2.1.1 (Weighted Degree)**.** Every weighted degree δ on the polynomial ring $A := \mathbb{K}[x_1, \ldots, x_n]$ is a semidegree. In this case there is of course a unique homomorphism $\iota : \operatorname{gr} A^\delta \to \mathbb{K}[x_1, \ldots, x_n]$ commuting with maps $A \ni f \mapsto [(f)_{\delta(f)}] \in \operatorname{gr} A^\delta$ and $A \ni f \mapsto \mathfrak{L}_\delta(f) \in \mathbb{K}[x_1, \ldots, x_n]$, where $[(f)_{\delta(f)}]$ is the equivalence class of $(f)_{\delta(f)}$ in $\operatorname{gr} A^\delta$ and $\mathfrak{L}_\delta(f)$ is the leading weighted homogeneous form of f corresponding to the weighted degree δ. Moreover, a straightforward calculation shows that ι is a graded $\mathbb{K}$-algebra isomorphism of $\operatorname{gr} A^\delta$ and $\mathbb{K}[x_1, \ldots, x_n]$. If $d_i := \delta(x_i)$ is positive for each i, then δ is a complete semidegree and the corresponding completion of $\mathbb{K}^n$ is the weighted projective space $\mathbb{P}^n(\mathbb{K}; 1, d_1, \ldots, d_n)$.

Example 2.1.2 (An iterated semidegree)**.** Let $X := \mathbb{K}^2$, ring $A := \mathbb{K}[x_1, x_2]$ and filtration $\mathcal{F} := \{F_d : d \ge 0\}$ on A be: $F_0 := \mathbb{K}$, $F_1 := \mathbb{K}\langle 1, x_1^2 - x_2^3 \rangle$, $F_2 := (F_1)^2 + \mathbb{K}\langle x_2 \rangle$, $F_3 := F_1 F_2 + \mathbb{K}\langle x_1 \rangle$ and $F_d := \sum_{j=1}^{d-1} F_j F_{d-j}$ for $d \ge 4$. We claim that function $\delta := \delta_{\mathcal{F}}$ is a semidegree. Indeed,

$$A^\delta = \mathbb{K}[(1)_1, (x_1)_3, (x_2)_2, (x_1^2 - x_2^3)_1] \cong \mathbb{K}[X_1, X_2, Y, Z]/\langle YZ^5 - X_1^2 + X_2^3 \rangle \ ,$$

where the last isomorphism is induced by a $\mathbb{K}$-algebra homomorphism which sends $X_1 \mapsto (x_1)_3$, $X_2 \mapsto (x_2)_2$, $Y \mapsto (x_1^2 - x_2^3)_1$ and $Z \mapsto (1)_1$. The inverse image of the ideal

$I := \langle (1)_1 \rangle$ of A^δ under this isomorphism coincides with ideal $\langle Z, YZ^5 - X_1^2 + X_2^3 \rangle = \langle Z, X_1^2 - X_2^3 \rangle$. Since the latter is a prime ideal of $\mathbb{K}[X_1, X_2, Y, Z]$, it follows that I is a prime ideal of A^δ as well. Then δ is a semidegree according to theorem 2.2.1. The isomorphism constructed above induces a closed embedding of X^δ onto hypersurface $V(YZ^5 - X_1^2 + X_2^3) \subseteq \mathbf{W}\mathbb{P}$, where $\mathbf{W}\mathbb{P}$ is the weighted projective space $\mathbb{P}^3(\mathbb{K}; 1, 3, 2, 1)$ with (weighted homogeneous) coordinates $[Z : X_1 : X_2 : Y]$. Constructed semidegree δ is an example of an *iterated semidegree* (introduced in section 3.2).

By definition a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is *integral* provided its vertices are in $\mathbb{Z}^n$.

Example 2.1.3 (Subdegrees determined by integral polytopes)**.** Let X be the n-torus $(\mathbb{K}^*)^n$ and $A := \mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ be its coordinate ring. Let $\mathcal{P}$ be a convex rational polytope (i.e. a convex polytope in $\mathbb{R}^n$ with vertices in $\mathbb{Q}^n$) of dimension n containing origin in its interior. Define a function $\delta' : A \setminus \{0\} \to \mathbb{Q}_+$ as follows:

$$\delta'(x^\alpha) := \inf\{r \in \mathbb{Q}_+ : \alpha \in r\mathcal{P}\},$$
$$\delta'(\sum a_\alpha x^\alpha) := \max_{a_\alpha \neq 0} \delta'(x^\alpha)$$

Claim. *There is $k \in \mathbb{N}$ such that $k\delta'$ is a complete subdegree.*

Proof. For each facet $\mathcal{Q}$ of $\mathcal{P}$, let $\omega_\mathcal{Q}$ be the smallest 'outward pointing' integral vector normal to $\mathcal{Q}$ and let $c_\mathcal{Q} = \langle \omega_\mathcal{Q}, \alpha \rangle$, where α is any element of the hyperplane that contains $\mathcal{Q}$. Since $\mathcal{P}$ is rational, in each $\mathcal{Q}$ there is an α with rational coordinates, and therefore each $c_\mathcal{Q}$ is a positive rational number. Let $\delta'_\mathcal{Q}$ be the $\mathbb{Q}$-valued weighted degree on A given by:

$$\delta'_\mathcal{Q}(\sum a_\alpha x^\alpha) := \max_{a_\alpha \neq 0} \frac{\langle \omega_\mathcal{Q}, \alpha \rangle}{c_\mathcal{Q}}.$$

For each $r \in \mathbb{R}_+$, $r\mathcal{P} = \{\beta \in \mathbb{R}^n : \langle \omega_\mathcal{Q}, \beta \rangle \leq rc_\mathcal{Q}$ for every facet $\mathcal{Q}$ of $\mathcal{P}\}$. It follows that for each $\alpha \in \mathbb{R}^n$,

$$\begin{aligned}
\delta'(x^\alpha) &:= \inf\{r \in \mathbb{Q}_+ : \alpha \in r\mathcal{P}\}, \\
&= \inf\{r \in \mathbb{Q}_+ : \langle \omega_{\mathcal{Q}}, \alpha \rangle \leq r c_{\mathcal{Q}} \text{ for every facet } \mathcal{Q} \text{ of } \mathcal{P}\} \\
&= \inf\{r \in \mathbb{Q}_+ : \delta'_{\mathcal{Q}}(x^\alpha) \leq r \text{ for every facet } \mathcal{Q} \text{ of } \mathcal{P}\} \\
&= \max\{\delta'_{\mathcal{Q}}(x^\alpha) : \mathcal{Q} \text{ is a facet of } \mathcal{P}\}.
\end{aligned}$$

Let $k \in \mathbb{N}$ be such that $k/c_{\mathcal{Q}}$ is an integer for each $\mathcal{Q}$. Then $k\delta'_{\mathcal{Q}}$ is an integer valued semidegree for each $\mathcal{Q}$ and hence $k\delta'$ is a subdegree.

We claim that $A^{k\delta'}$ is finitely generated. Indeed, since $\mathcal{P}$ is rational, there is an $l \in \mathbb{N}$ such that $l\mathcal{P}$ is *integral*. Identify $\mathbb{R}^n$ with the hyperplane $x_{n+1} = 1$ in $\mathbb{R}^{n+1}$. Let $\mathcal{C}$ be the cone in $\mathbb{R}^{n+1}$ over $l\mathcal{P} \subseteq \mathbb{R}^n$ (vertex of $\mathcal{C}$ being the origin). Then, with $d := kl$, the d-th truncated ring of $A^{k\delta'}$ is

$$\begin{aligned}
(A^{k\delta'})^{[d]} &:= \bigoplus_{m \geq 0} \{f \in A : k\delta'(f) \leq klm\} \\
&= \bigoplus_{m \geq 0} \{f \in A : \delta'(f) \leq lm\} \\
&= \bigoplus_{m \geq 0} \mathbb{K}\text{-span}\{x^\alpha : \alpha \in lm\mathcal{P}\} \\
&= \bigoplus_{m \geq 0} \mathbb{K}\text{-span}\{x^\alpha : (\alpha, m) \in \mathcal{C} \cap \mathbb{Z}^{n+1}\}.
\end{aligned}$$

By Gordan's lemma [9, Proposition 1, Section 1.2] the semigroup $\mathcal{C} \cap \mathbb{Z}^{n+1}$ is finitely generated. It follows that $(A^{k\delta'})^{[kl]}$ is a finitely generated $\mathbb{K}$-algebra. But ring $A^{k\delta'}$ is integral over ring $(A^{k\delta'})^{[kl]}$, since for every $(f)_e \in A^{k\delta'}$, $e \in \mathbb{Z}_+$, $((f)_e)^{kl} \in (A^{k\delta'})^{[kl]}$. Therefore $A^{k\delta'}$ is a $(A^{k\delta'})^{[kl]}$-submodule of the integral closure of $(A^{k\delta'})^{[kl]}$. Since the latter is a finite $(A^{k\delta'})^{[kl]}$-module, it follows that $A^{k\delta'}$ is also a finite $(A^{k\delta'})^{[kl]}$-module and hence is a finitely generated algebra over $\mathbb{K}$. □

Let $\delta := k\delta'$, where k is an integer as in the above claim. Recall from section 1.0.3 that for each $d > 0$, a set of homogeneous generators of $(A^\delta)^{[d]}$ gives rise to the d-uple embedding of X^δ into a weighted projective space. Let d be a positive integer such that $(A^\delta)^{[d]}$ is generated as a $\mathbb{K}$-algebra by elements in the d-th graded component of A^δ, or equivalently the $\mathbb{K}$-span of monomials x^α such that $\alpha \in \frac{d}{k}\mathcal{P}$. Then the image of the d-uple embedding of X^δ is the closure (in the appropriate dimensional projective space) of the image of $(\mathbb{K}^*)^n$ under the map whose components are the monomials with exponents in $\frac{d}{k}\mathcal{P}$. For $\mathbb{K} = \mathbb{C}$, it follows by theorem 2.0.2 that X^δ is isomorphic via the d-uple embedding to toric variety $X_{\mathcal{P}}$.

2.2 Properties of Subdegree

2.2.1 Uniqueness of minimal presentations of subdegrees by semidegrees and a characterization via absence of nilpotents in their graded rings.

Recall that an ideal $\mathfrak{q}$ of a ring R is called *primary* if $\mathfrak{q} \neq R$ and for all $f, g \in R$, $fg \in \mathfrak{q} \Rightarrow$ either $f \in \mathfrak{q}$ or $g^k \in \mathfrak{q}$ for some $k \geq 1$. A *primary decomposition* of an ideal $\mathfrak{a}$ of R is an expression of $\mathfrak{a}$ as a finite intersection of primary ideals. $\mathfrak{a}$ is called *decomposable* if it has a primary decomposition. Below we use the following commonly used notation: for any ideal $\mathfrak{a}$ of R and any element $f \in R$, $(\mathfrak{a} : f)$ is the ideal of R defined by: $(\mathfrak{a} : f) := \{g \in R : fg \in \mathfrak{a}\}$.

Proposition 2.2.0 (See [8, Chapter 3]). *$\mathfrak{a}$ is a decomposable radical ideal if and only if it is a finite intersection of* unique *prime ideals of the form $(\mathfrak{a} : f)$ for $f \in R \setminus \mathfrak{a}$. If R is a graded ring and $\mathfrak{a}$ is a homogeneous ideal, then the elements f in the preceding statement are also homogeneous. If R is Noetherian, then every ideal of R is decomposable.*

Theorem 2.2.1 (see [25, Theroem 2.1] and [26, Theorem 2.2.1]). *Let δ be a degree like function on A and let I be the ideal of A^δ generated by $(1)_1$.*

1. *δ is a semidegree if and only if I is a prime ideal.*

2. δ is a subdegree if and only if I is a decomposable radical ideal.

3. If A^δ is Noetherian, then δ is a subdegree if and only if I is a radical ideal.

Proof. I is a prime ideal iff $(f)_d, (g)_e \notin I$ implies $(f)_d(g)_e = (fg)_{d+e} \notin I$. Since $(f)_d \notin I$ iff $\delta(f) = d$, the preceding condition is equivalent to the condition that whenever $\delta(f) = d$ and $\delta(g) = e$, then $\delta(fg) = d + e$. The latter is the defining property of a semidegree, and assertion 1 is proved.

Due to proposition 2.2.0, assertion 3 follows from assertion 2. So we proceed with proving assertion 2. The following lemma reformulates the property of I being radical:

Lemma 2.2.1.1. *I is radical iff $\delta(f^k) = k\delta(f)$ for all $f \in A$ and $k \geq 0$.*

Proof. I is radical iff $f \in A$ and $(f)_d \in A^\delta \setminus I$ imply $((f)_d)^k = (f^k)_{dk} \in A^\delta \setminus I$ for all $k \geq 0$. Since $(f)_d \in A^\delta \setminus I$ iff $\delta(f) = d$, it follows that I is radical iff $\delta(f^k) = k\delta(f)$ for all $f \in A$ and $k \geq 0$. □

At first assume δ is a subdegree. Let $\delta = \max_{i=1}^N \delta_i$ be a minimal presentation for δ. Since for all $f \in A$ and $k \geq 0$, $\delta_i(f^k) = k\delta_i(f)$ for each i, it follows that $\delta(f^k) = k\delta(f)$. Therefore, by lemma 2.2.1.1, I is radical.

It remains to show for the 'only if' implication that ideal I is decomposable, i.e. it is the intersection of *finitely* many primes. Recall that temporarily (up to Theorem 2.2.11) we allow degree like functions (including semi- and subdegrees) to have $-\infty$ as a value. For each i with $0 \leq i \leq N$, fix an $f_i \in A$ such that $\delta_i(f_i) > \delta_j(f_i)$ for all $j \neq i$. Then, in particular, $\delta_i(f_i) \in \mathbb{Z}$. Let $d_i := \delta(f_i) = \delta_i(f_i)$. That I is decomposable is a consequence of assertion 2 of the following lemma.

Lemma 2.2.1.2.

(1) For each i, $1 \leq i \leq N$, $(I : (f_i)_{d_i})$ is a homogeneous ideal, and the homogeneous elements of $(I : (f_i)_{d_i})$ are precisely the elements of $L_i := \{(f)_d : f \in A, d > \delta_i(f)\}$.

(2) For each i, $(I : (f_i)_{d_i})$ is a distinct minimal prime ideal of A^δ containing I. In particular, $(f_i)_{d_i} \in (\bigcap_{j \neq i}(I : (f_j)_{d_j})) \setminus (I : (f_i)_{d_i})$ for each i. Moreover,

$$\bigcap_{i=1}^{N}(I : (f_i)_{d_i}) = I. \tag{2.2}$$

Proof. We first prove assertion 1. Fix an i, $1 \leq i \leq N$. Since $(f_i)_{d_i}$ is a homogeneous element in A^δ and I is a homogeneous ideal of A^δ, it follows that $(I : (f_i)_{d_i})$ is also a homogeneous ideal of A^δ. Let $(f)_d$ be an arbitrary homogeneous element of $(I : (f_i)_{d_i})$. If $\delta(f) < d$, then $\delta_i(f) \leq \delta(f) < d$ and $(f)_d \in L_i$. So assume $\delta(f) = d$. Since $(f)_d(f_i)_{d_i} = (ff_i)_{d+d_i} \in I$, it follows that $\delta(ff_i) < d + d_i$. But then $\delta_i(ff_i) = \delta_i(f) + \delta_i(f_i) < d + d_i$, which implies that $\delta_i(f) < d = \delta(f)$, and thus $(f)_d \in L_i$. To summarize, all homogeneous elements of $(I : (f_i)_{d_i})$ belong to L_i.

Now let $(f)_d$ be an arbitrary element of L_i. If $d > \delta(f)$, then $(f)_d \in I \subseteq (I : (f_i)_{d_i})$. So assume $d = \delta(f)$. Then $\delta_i(f) < d = \delta(f)$, and thus $\delta_i(ff_i) = \delta_i(f) + \delta_i(f_i) < d + d_i$. Also, for each $j \neq i$, $\delta_j(f_i) < d_i$, so that $\delta_j(ff_i) = \delta_j(f) + \delta_j(f_i) < d + d_i$. It follows that $\delta(ff_i) < d + d_i$ and $(f)_d(f_i)_{d_i} = (ff_i)_{d+d_i} \in I$, i.e. $(f)_d \in (I : (f_i)_{d_i})$, which proves inclusion $L_i \subseteq (I : (f_i)_{d_i})$ and completes the proof of assertion 1.

We now show that $(I : (f_i)_{d_i})$ is prime. Let $(g_1)_{e_1}, (g_2)_{e_2}$ be homogeneous elements of A^δ such that $(g_1)_{e_1}(g_2)_{e_2} \in (I : (f_i)_{d_i})$. Due to assertion 1, $(g_1g_2)_{e_1+e_2} = (g_1)_{e_1}(g_2)_{e_2} \in L_i$, which means that $\delta_i(g_1g_2) = \delta_i(g_1) + \delta_i(g_2) < e_1 + e_2$. Since $e_j \geq \delta(g_j) \geq \delta_i(g_j)$ for each j, it follows that there is j, $j = 1$ or 2, such that $\delta_i(g_j) < e_j$. Then assertion 1 implies that $(g_j)_{e_j} \in (I : (f_i)_{d_i})$, i.e. $(I : (f_i)_{d_i})$ is a prime ideal. Next we show that $(I : (f_i)_{d_i})$ is a minimal prime ideal of A^δ containing ideal I.

Indeed, if $I \subseteq \mathfrak{p} \subseteq (I : (f_i)_{d_i})$ for a prime ideal p of A^δ, and some $i \leq N$, then $(f_i)_{d_i} \notin \mathfrak{p}$ (since assertion 1 implies $(f_i)_{d_i} \notin (I : (f_i)_{d_i})$). But if $(g)_e \in (I : (f_i)_{d_i})$, then $(g)_e(f_i)_{d_i} \in I \subseteq \mathfrak{p}$ and it follows that $(g)_e \in \mathfrak{p}$. Hence $(I : (f_i)_{d_i}) \subseteq \mathfrak{p}$, and therefore $(I : (f_i)_{d_i}) = \mathfrak{p}$, as required.

To see that the ideals $(I : (f_i)_{d_i})$ are distinct, note that $(f_i)_{d_i} \in (\bigcap_{j \neq i} L_j) \setminus L_i$ for

$1 \leq i \leq N$, which implies due to assertion 1 that $(f_i)_{d_i} \in (\bigcap_{j \neq i}(I : (f_j)_{d_j})) \setminus (I : (f_i)_{d_i})$ for every i.

Finally, pick any homogeneous $(g)_e \in \bigcap(I : (f_i)_{d_i})$. Then by assertion 1, for each i, $\delta_i(g) < e$. Therefore $\delta(g) = \max_i \delta_i(g) < e$, and hence $(g)_e \in I$. It follows that $\bigcap(I : (f_i)_{d_i}) = I$, which concludes the proof of the lemma. □

We now return to the proof of theorem 2.2.1 and prove the 'if' implication of assertion 2. Assume I is a decomposable radical ideal of A^δ. We have to show that δ is a subdegree. According to proposition 2.2.0 there exist $(f_1)_{d_1}, \ldots, (f_N)_{d_N} \in A^\delta \setminus I$ such that $(I : (f_i)_{d_i})$ is a prime ideal containing I for each i and $I = \bigcap_{i=1}^{N}(I : (f_i)_{d_i})$. Since $(f_i)_{d_i} \notin I$, it follows that $\delta(f_i) = d_i \in \mathbb{Z}$ for each i. For each $i = 1, \ldots, N$, let $\delta_i : A \to \mathbb{Z} \cup \{-\infty\}$ be defined as follows:

$$\delta_i(f) := \lim_{k \to \infty} \delta((f_i)^k f) - \delta((f_i)^k). \tag{2.3}$$

We first show that δ_i is well defined for each i. Indeed, fix an i, $1 \leq i \leq N$. Let $f \in A$ and $k \geq 1$. Since $(f_i)_{d_i} \notin I$ and I is radical, it follows that $((f_i)_{d_i})^k = ((f_i)^k)_{kd_i} \notin I$, so that $\delta((f_i)^k) = kd_i$. Therefore,

$$\begin{aligned} \delta((f_i)^{k+1} f) - \delta((f_i)^{k+1}) &\leq \delta((f_i)^k f) + \delta(f_i) - \delta((f_i)^{k+1}) \\ &= \delta((f_i)^k f) + d_i - (k+1)d_i \\ &= \delta((f_i)^k f) - kd_i \\ &= \delta((f_i)^k f) - \delta((f_i)^k). \end{aligned} \tag{2.4}$$

It follows that $\delta_i(f)$ is a well defined element in $\mathbb{Z} \cup \{-\infty\}$.

Multiplicativity: Let $f, g \in A$. We now show that $\delta_i(fg) = \delta_i(f) + \delta_i(g)$, splitting the proof into the following three cases:

Case 1: $\delta_i(f)$ and $\delta_i(g)$ are integers. Note that $\delta_i(f) = d \in \mathbb{Z}$

$$\begin{aligned}
&\iff \exists k_f \in \mathbb{N} \text{ such that } \delta((f_i)^{k+1}f) - \delta((f_i)^{k+1}) = d \text{ for all } k \geq k_f,\\
&\iff \exists k_f \in \mathbb{N} \text{ such that } \delta((f_i)^{k+1}f) = (k+1)d_i + d \text{ for all } k \geq k_f, \qquad (2.5)\\
&\iff \exists k_f \in \mathbb{N} \text{ such that } ((f_i)^k f)_{kd_i+d}(f_i)_{d_i} = ((f_i)^{k+1}f)_{(k+1)d_i+d} \notin I \text{ for all } k \geq k_f,\\
&\iff \exists k_f \in \mathbb{N} \text{ such that } ((f_i)^k f)_{kd_i+d} \notin (I : (f_i)_{d_i}) \text{ for all } k \geq k_f.
\end{aligned}$$

Therefore, if both $\delta_i(f)$ and $\delta_i(g)$ are integers, then there is a $k' := \max\{k_f, k_g\} \in \mathbb{N}$ such that for all $k \geq k'$, $((f_i)^k f)_{kd_i+\delta_i(f)}$ and $((f_i)^k g)_{kd_i+\delta_i(g)}$ do not lie in $(I : (f_i)_{d_i})$. Since $(I : (f_i)_{d_i})$ is a prime ideal, it follows that for all $k \geq k'$, $((f_i)^k f)_{kd_i+\delta_i(f)}((f_i)^k g)_{kd_i+\delta_i(g)} = ((f_i)^{2k} fg)_{2kd_i+\delta_i(f)+\delta_i(g)} \notin (I : (f_i)_{d_i})$. Therefore $\delta_i(fg) = \delta_i(f) + \delta_i(g) \in \mathbb{Z}$ due to (2.5), as required.

If case 1 does not take place, we may w.l.o.g. assume that $\delta_i(f) = -\infty$.

Case 2: $\delta_i(f) = -\infty$ and $\delta(g) \in \mathbb{Z}$. Note that for all $h \in A$, $\delta_i(h) = -\infty$ if and only if for every $n \in \mathbb{Z}$ there is a $k' \in \mathbb{N}$ such that for all $k \geq k'$, $\delta((f_i)^k h) - \delta((f_i)^k) < n$. Pick an $n \in \mathbb{Z}$. Since $\delta_i(f) = -\infty$, we can choose $k' \in \mathbb{N}$ such that for all $k \geq k'$, $\delta((f_i)^k f) - \delta((f_i)^k) < n - \delta(g)$. Then $\delta((f_i)^k fg) - \delta((f_i)^k) \leq \delta((f_i)^k f) + \delta(g) - \delta((f_i)^k) < n$ for all $k \geq k'$. Therefore $\delta(fg) = -\infty = \delta(f) + \delta(g)$.

Case 3: $\delta_i(f) = -\infty$ and $\delta(g) = -\infty$. If $\delta(g)$ is $-\infty$, then for all $h \in A$, $\delta(gh) \leq \delta(g) + \delta(h) \leq -\infty$, and hence $\delta(gh) = -\infty$. But then $\delta(f_i^k fg) = -\infty$ for all k and therefore $\delta_i(fg) = -\infty = \delta_i(f) + \delta_i(g)$.

The above 3 cases cover all possible options and therefore we proved that each δ_i is multiplicative.

Additivity: Let $f, g \in A$. We first show that $\delta_i(f + g) \leq \max\{\delta_i(f), \delta_i(g)\}$. Since δ is a degree like function, it follows that for all $k \in \mathbb{N}$,

$$\delta(f_i^k(f + g)) - \delta(f_i^k) \leq \max\{\delta(f_i^k f) - \delta(f_i^k), \delta(f_i^k g) - \delta(f_i^k)\}.$$

The required inequality then follows from the following simple observation: if $\{a_k\}$, $\{b_k\}$

and $\{c_k\}$ are decreasing sequences of real numbers such that $a_k \leq \max\{b_k, c_k\}$ for each k, then $\lim a_k \leq \max\{\lim b_k, \lim c_k\}$.

Next assume that $\delta_i(f+g) < \max\{\delta_i(f), \delta_i(g)\}$. We will show, as required, that $\delta_i(f) = \delta_i(g)$. Assume otherwise, say $\delta_i(f) < \delta_i(g)$. Then $\delta_i(g)$ is an integer and it follows by definition of δ_i that there exists $k' \in \mathbb{N}$ such that for all $k \geq k'$, $\delta(f_i^k f) - \delta(f_i^k) < \delta_i(g) = \delta(f_i^k g) - \delta(f_i^k)$. Consequently, for all $k \geq k'$, $\delta(f_i^k g) > \delta(f_i^k f)$, and therefore due to property 2 of the definition of degree like functions, $\delta(f_i^k(f+g)) - \delta(f_i^k) = \delta(f_i^k g) - \delta(f_i^k) = \delta_i(g)$. It follows that $\delta_i(f+g) = \delta_i(g)$ contrary to our assumption. Therefore $\delta_i(f) = \delta_i(g)$ and it concludes the proof that δ_i is a semidegree.

The only remaining assertion to prove is $\delta = \max_{i=1}^N \delta_i$. First of all note that for all $f \in A$, and $1 \leq i \leq N$, $\delta(f_i f) \leq \delta(f_i) + \delta(f)$, so that $\delta(f) \geq \delta(f_i f) - \delta(f_i) \geq \delta_i(f)$. It follows that $\delta \geq \max_{i=1}^N \delta_i$.

Now pick any $f \in A$. If $\delta(f) = -\infty$, then for all i and for all $k \geq 0$, $\delta((f_i)^k f) = -\infty$, and therefore $\delta_i(f) = -\infty = \delta(f)$.

It remains to consider the case that $\delta(f) \in \mathbb{Z}$. Since $(f)_{\delta(f)} \notin I$, it follows that there is an i such that $(f)_{\delta(f)} \notin (I : (f_i)_{d_i})$. Note that $(f_i)_{d_i}$ also does not lie in $(I : (f_i)_{d_i})$, for otherwise $((f_i)_{d_i})^2$ would be an element of I and this would imply that $(f_i)_{d_i} \in I$ (since I is a radical ideal), contradicting our choice of $(f_i)_{d_i}$. Thus neither of $(f)_{\delta(f)}$ and $(f_i)_{d_i}$ is an element of $(I : (f_i)_{d_i})$. Since $(I : (f_i)_{d_i})$ is a prime ideal, it follows that for all $k \geq 0$, $((f_i)_{d_i})^k (f)_{\delta(f)} = ((f_i)^k f)_{kd_i + \delta(f)} \notin (I : (f_i)_{d_i})$. Then according to (2.5), $\delta_i(f) = \delta(f)$, which proves $\delta = \max_{i=1}^N \delta_i$ and that δ is indeed a subdegree. □

Corollary 2.2.2. *Assume A^δ is Noetherian. Then δ is a subdegree on A iff $\delta(f^k) = k\delta(f)$ for all $f \in A$ and $k \geq 0$.*

Proof. This claim is a straightforward consequence of assertion 3 of theorem 2.2.1 and lemma 2.2.1.1. □

Proposition 2.2.3. *Let δ be a subdegree on A. Semidegrees $\delta_1, \ldots, \delta_N$ of a minimal presentation $\delta = \max_{1 \leq i \leq N} \delta_i$ of δ are unique.*

Proof. Assume semidegrees $\delta_1, \dots, \delta_N$ and $\delta'_1, \dots, \delta'_{N'}$ provide two distinct minimal presentations $\delta = \max_{1 \le i \le N} \delta_i$ and $\delta = \max_{1 \le i \le N'} \delta'_i$ of δ. As in the proof of theorem 2.2.1, there exist $f_1, \dots, f_N$ and $f'_1, \dots, f'_{N'} \in A$ such that

$$d_i := \delta_i(f_i) > \delta_j(f_i) \text{ for } 1 \le j \ne i \le N, \tag{2.6}$$

$$d'_i := \delta'_i(f'_i) > \delta'_j(f'_i) \text{ for } 1 \le j \ne i \le N'. \tag{2.6$'$}$$

Then d_i and d'_i are integers, and by lemma 2.2.1.2,

$$I = \bigcap_{i=1}^{N} (I : (f_i)_{d_i}) \quad \text{and} \quad I = \bigcap_{i=1}^{N'} (I : (f'_i)_{d'_i})$$

are *unique* minimal primary decompositions of I, with ideal I in A^δ being generated by $(1)_1$. The latter uniqueness implies that $N = N'$ and, after an appropriate re-indexing of δ'_i's, that ideals $(I : (f_i)_{d_i})$ and $(I : (f'_i)_{d'_i})$ coincide for all $i \le N$.

Fix an i, $1 \le i \le N$. According to assertion 2 of lemma 2.2.1.2, elements

$$(f'_i)_{d'_i} \in (\bigcap_{j \ne i} (I : (f'_j)_{d'_j})) \setminus (I : (f'_i)_{d'_i}) = (\bigcap_{j \ne i} (I : (f_j)_{d_j})) \setminus (I : (f_i)_{d_i}) \, .$$

Therefore assertion 1 of lemma 2.2.1.2 implies that $(f'_i)_{d'_i} \in (\bigcap_{j \ne i} L_j) \setminus L_i$, where L_j's are as in lemma 2.2.1.2. It follows that $\delta_i(f'_i) = \delta(f'_i) = \delta'_i(f'_i)$, and $\delta_j(f'_i) < \delta_i(f'_i)$ for all $j \ne i$. The latter implies property (2.6) with f'_i's replacing f_i's. Since f_i's were assumed to be arbitrary elements in A such that (2.6) is true, we may assume without loss of generality that $f_i = f'_i$ for each i. It follows that $\delta_i = \delta'_i$ for all i by making use of

Lemma 2.2.4. *If $f_1, \dots, f_N \in A$ satisfy (2.6), then $\delta_i(f) = \lim_{k \to \infty} \delta((f_i)^k f) - \delta((f_i)^k)$ for all $f \in A$.*

Proof. Fix an i, $1 \le i \le N$. Let us write $\tilde{\delta}_i(f) := \lim_{k \to \infty} \delta((f_i)^k f) - \delta((f_i)^k)$ for all $f \in A$. It follows by the arguments in the proof of theorem 2.2.1 that $\tilde{\delta}_i$ is a well defined semidegree on A. Moreover, for all $k \ge 0$ and all $f \in A$, $\delta((f_i)^k f) - \delta((f_i)^k) \ge \delta_i((f_i)^k f) - \delta((f_i)^k) = \delta_i(f)$, so that $\tilde{\delta}_i \ge \delta_i$. To see the opposite inequality, let $f \in A$ and $d \in \mathbb{Z}$ be such that $d \ge \delta_i(f)$. Then $kd_i + d \ge \delta_i((f_i)^k f)$ for all k. Moreover, (2.6) implies that for sufficiently large k, $kd_i + d > k\delta_j(f_i) + \delta_j(f) = \delta_j((f_i)^k f)$ for all $j \neq i$. It follows that for sufficiently large k, $\delta((f_i)^k f) \leq kd_i + d$, and hence $\delta((f_i)^k f) - \delta((f_i)^k) \leq d$. Since $\{\delta((f_i)^k f) - \delta((f_i)^k)\}$ is a decreasing sequence due to (2.4), it follows that $\tilde{\delta}_i(f) \leq d$. With $d = \delta_i(f)$ when $\delta_i(f) \in \mathbb{Z}$, and otherwise letting d converge to $\delta_i(f)$ from above, we see that $\tilde{\delta}_i(f) \leq \delta_i(f)$. It follows that $\delta_i(f) = \tilde{\delta}_i(f)$, which completes the proof of proposition 2.2.3. □

Corollary 2.2.5. *Let δ be a subdegree on A and $\delta = \max_{i=1}^N \delta_i$ be its minimal presentation. Then $X_\infty := \operatorname{Proj} A^\delta \setminus \operatorname{Spec} A$ has exactly N irreducible components.*

Proof. By theorem 1.1.4, $X_\infty = V(I) \subseteq \operatorname{Proj} A^\delta$, where I is the ideal in A^δ generated by $(1)_1$. Let $f_1, \ldots, f_N$, $d_1, \ldots, d_N$ be as in (2.6). Then $\mathfrak{p}_i := (I : (f_i)_{d_i})$ is a minimal prime ideal containing I and $I = \bigcap_{i=1}^N \mathfrak{p}_i$ is the prime decomposition of I. It follows that $X_\infty = V(I) = \bigcup_{i=1}^N V(\mathfrak{p}_i)$ is the decomposition of X_∞ into irreducible components. □

Example 2.2.6. Let $X = (\mathbb{K}^*)^n$ and let $\mathcal{P}$ be a convex integral polytope of dimension n containing the origin in its interior. We saw in example 2.1.3 that the toric variety $X_\mathcal{P}$ is isomorphic to X^δ for a subdegree δ on $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. Moreover, the minimal presentation of δ is of the form: $\delta = \max\{\delta_\mathcal{Q} : \mathcal{Q} \text{ is a facet of } \mathcal{P}\}$. It follows by corollary 2.2.5 that the components of $X_\mathcal{P} \setminus (\mathbb{K}^*)^n$ are in a one-to-one correspondence with the facets of $\mathcal{P}$ (cf. theorem 2.0.2).

A subdegree on an integrally closed domain has an additional property:

Proposition 2.2.7. *If A is an integrally closed domain and δ is a subdegree on A, then A^δ is also integrally closed.*

Proof. In (1.1) we introduced an isomorphism $A^\delta \cong \bigoplus_{i\in\mathbb{Z}} F_i t^i \subseteq A[t, t^{-1}]$, with $(1)_1$ being mapped to t. Since A is an integrally closed domain, it follows that $A[t, t^{-1}]$ is also integrally closed [1, Exercise 5.9]. So it suffices to show that A^δ is integrally closed in

$A[t,t^{-1}]$. Pick $f = \sum_{i=q}^{r} f_i t^i \in A[t,t^{-1}]$ integral over A^δ, where $f_i \in A$ for each i. Then f satisfies an equation of the form

$$T^m + G_1 T^{m-1} + \cdots + G_m = 0 \tag{2.7}$$

for $G_1, \ldots, G_m \in A^\delta$. Taking the highest degree terms in t, we see that $f_r t^r$ is integral over f. Replacing f by $f - f_r t^r$ and repeating the procedure, it follows that each $f_i t^i$ is integral over A^δ. Therefore it suffices to show that if $ft^k \in A[t,t^{-1}]$ is integral over A^δ for some $f \in A$, then $ft^k \in A^\delta$. To that end, let $f \in A$ be such that ft^k satisfies an equation of form (2.7). Comparing the coefficients at t^{km} in that equation, we may assume that $G_i = g_i t^{ik}$ for some $g_i \in A$. Since $g_i t^{ik} \in A^\delta$, it follows that $\delta(g_i) \leq ik$.

Assume contrary to the assertion of the proposition that $ft^k \notin A^\delta$. Then $d := \delta(f) > k$. Set $T = ft^k$, $G_i = g_i t^{ik}$ and $t = 1$ in (2.7). It follows that $f^m = -\sum_{i=1}^m g_i f^{m-i}$ in A. For each $i \geq 1$, $\delta(g_i f^{m-i}) \leq \delta(g_i) + \delta(f^{m-i}) \leq ik + (m-i)d < id + (m-i)d = md$. Therefore $\delta(f^m) = \delta(-\sum_{i=1}^m g_i f^{m-i}) \leq \max_{i=1}^m \delta(g_i f^{m-i}) < md$. On the other hand, since δ is a subdegree, $\delta(f^m) = m\delta(f) = md$. The contradiction we arrived at proves $ft^k \in A^\delta$, as required. □

Corollary 2.2.8. *Let X be a normal affine variety with A being the ring of regular functions on X and δ being a complete subdegree on A. Then* $\operatorname{Spec} A^\delta$ *and* $\operatorname{Proj} A^\delta$ *are normal varieties as well.* □

Example 2.2.9. In this example rings $A := \mathbb{K}[x]$ and A^δ are integrally closed, where degree like function δ on $\mathbb{K}[x]$ is defined by

$$\delta(x^k) := \begin{cases} 3k/2 & \text{if } k \text{ is even,} \\ 3(k-1)/2 + 2 & \text{if } k \text{ is odd.} \end{cases}$$

We claim that degree like function δ is not a subdegree.

Indeed, $\mathbb{K}[x]^\delta = \mathbb{K}[(1)_1, (x)_2, (x^2)_3] \cong \mathbb{K}[x,y,z]/\langle x^2 - yz \rangle$, where the isomorphism is induced by the $\mathbb{K}$-algebra homomorphism $\mathbb{K}[x,y,z] \to \mathbb{K}[x]^\delta$ which sends $x \mapsto (x)_2$,

$y \mapsto (x^2)_3$ and $z \mapsto (1)_1$. If $\mathbb{K}$ is not of characteristic 2, then $\mathbb{K}[x]^\delta$ is integrally closed [14, Exercise II.6.4]. On the other hand, $\delta(x^2) = 3 < 4 = 2\delta(x)$ and it follows by making use of corollary 2.2.2 that δ is not a subdegree.

2.2.2 For finitely generated subdegrees, the semidegrees of the minimal presentation do not take $-\infty$ value.

In the proof of the next two theorems we make an essential use of the theory of *Rees valuation*. We first describe following [24, chapter XI] the relevant results of Rees (starting with a reminder of the notion of a *Krull domain*).

Definition. A domain B is a *Krull domain* iff

1. $B_\mathfrak{p}$ is a discrete valuation ring for all height one prime ideals $\mathfrak{p}$ of B, and
2. every non-zero principal ideal of B is the intersection of a finite number of primary ideals of height one.

Every normal Noetherian domain is a Krull domain [23, Section 41]. In particular, the integral closure of A^δ is a Krull domain provided that A^δ is finitely generated.

For an ideal I of a ring R define $\nu_I : R \to \mathbb{N} \cup \{\infty\}$ and $\bar{\nu}_I : R \to \mathbb{Q}_+ \cup \{\infty\}$ by:

$$\nu_I(x) := \sup\{m : x \in I^m\}, \text{ and}$$
$$\bar{\nu}_I(x) := \lim_{m\to\infty} \frac{\nu_I(x^m)}{m},$$

for all $x \in R$. Recall that the *integral closure* $\bar{J}$ of an ideal J of R is the ideal defined by: $\bar{J} := \{x \in R : x$ satisfies an equation of the form: $x^s + j_1x^{s-1} + \cdots + j_s = 0$ with $j_k \in J^k$ for all $k = 1, \ldots, s\}$. The following is due to Rees [24, Propositions 11.1 – 11.6]:

Theorem (Rees). *For any ring R and any ideal I of R, $\bar{\nu}_I$ is well defined. Assume R is a Noetherian domain. Then*

(1) there is a positive integer e such that for all $x \in R$, $\bar{\nu}_I(x) \in \frac{1}{e}\mathbb{N}$, and

(2) if $k \geq 0$ is an integer then $\bar{\nu}_I(x) \geq k$ if and only if $x \in \bar{I^k}$, where $\bar{I^k}$ is the integral closure of I^k in R.

(3) Assume in addition that I is a principal ideal generated by u and $\bar{R}$ is an integral extension of R which is a Krull domain. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height 1 prime ideals of $\bar{R}$ containing u. Then for all $x \in R$, $\bar{\nu}_I(x) = \min\{\frac{\nu_i(x)}{e_i} : i = 1, \ldots, r\}$, where for each $i = 1, \ldots, r$, ν_i is the valuation associated with the discrete valuation ring $\bar{R}_{\mathfrak{p}_i}$ and $e_i := \nu_i(u)$.

The following theorem gives a characterization of the semidegrees δ_i associated to a subdegree δ, provided that A^δ is finitely generated. In particular, it states that each δ_i is integer valued (which is not a priori obvious relying only on the limit definition of δ_i's in (2.3)). We start with the following

Lemma 2.2.10. *If A^δ is Noetherian, then $\delta(f) \in \mathbb{Z}$ for all $f \in A$, i.e. δ does* not *take the value $-\infty$.*

Proof. Recall that A is a domain and hence A^δ is also a domain by lemma 1.1.1. Let I be the ideal generated by $(1)_1$ in A^δ. Since $(1)_0 \notin I$, ideal I is a proper ideal of A^δ. It follows that if A^δ is Noetherian, then $\bigcap_{k=1}^\infty I^k = \emptyset$ [1, Corollary 10.18]. Assume contrary to the claim that $\delta(f) = -\infty$. Then $(f)_k \in A^\delta$ for all $k \in \mathbb{Z}$, and therefore $(f)_0 \in \bigcap_{k=1}^\infty I^k$ (since for each $k \geq 0$, $(f)_0 = (f)_{-k} \cdot ((1)_1)^k \in I^k$). It follows that $\bigcap_{k=1}^\infty I^k \neq \emptyset$, which is in contradiction with the previous conclusion, and therefore $\delta(f) \neq -\infty$ for all $f \in A$, as required. ☐

Theorem 2.2.11 (see [26, Theorem 2.2.7])**.** *Let δ be a finitely generated subdegree on A with a minimal presentation $\delta = \max_{1 \leq i \leq N} \delta_i$. Let B be any Krull domain which is also an integral extension of A^δ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height one primes of B containing $(1)_1$. For each j, $1 \leq j \leq r$, define a function $\hat{\delta}_j$ on $A \setminus \{0\}$ by*

$$\hat{\delta}_j(f) = \delta(f) - \frac{\nu_j((f)_{\delta(f)})}{e_j}$$

where ν_j is the discrete valuation of the discrete valuation ring $B_{\mathfrak{p}_j}$ and $e_j := \nu_j((1)_1)$. Then for each i, $1 \leq i \leq N$, $\delta_i \equiv \hat{\delta}_j$ for some j, $1 \leq j \leq r$. In particular, semidegrees δ_i are integer valued for all i.

Proof. Let I be the ideal generated by $(1)_1$ in A^δ and $\bar{\nu}_I$ be the Rees' valuation corresponding to I. Let $f \in A$. According to lemma 2.2.10, $\delta(f) \in \mathbb{Z}$. Since δ is a subdegree, $\delta(f^k) = k\delta(f)$ for all $k \geq 0$ (corollary 2.2.2) and hence $((f)_{\delta(f)})^k = (f^k)_{k\delta(f)} \notin I$ for all $k \geq 0$. It follows that $\bar{\nu}_I((f)_{\delta(f)}) = 0$. Now assertion 3 of Rees' theorem implies that $\min_{j=1}^r \frac{\nu_j((f)_{\delta(f)})}{e_j} = 0$, where $e_j := \nu_j((1)_1)$ for every j. Therefore $\delta(f) = \delta(f) - \min_{j=1}^r \frac{\nu_j((f)_{\delta(f)})}{e_j} = \max_{j=1}^r \hat{\delta}_j(f)$ for all $f \in A$. Next we show that for each j, function $e_j\hat{\delta}_j$ is a $\mathbb{Z}$-*valued* semidegree, which suffices to complete the proof of theorem 2.2.11 due to the uniqueness of the minimal presentation of subdegrees (proposition 2.2.3) applied to the minimal presentation $e\delta = \max_i e\delta_i$ and the presentation $e\delta = \max_j e\hat{\delta}_j$ with an appropriate $e \in \mathbb{Z}_+$ (e.g. $e = \prod_j e_j$).

Multiplicativity: Fix j, $1 \leq j \leq r$. Let $f, g \in A \setminus \{0\}$. First we show that $\hat{\delta}_j(fg) = \hat{\delta}_j(f) + \hat{\delta}_j(g)$. Let $\epsilon := \delta(f) + \delta(g) - \delta(fg)$. Then

$$\begin{aligned}
\hat{\delta}_j(fg) &= \delta(fg) - \frac{1}{e_j}\nu_j((fg)_{\delta(fg)}) = \delta(f) + \delta(g) - \frac{1}{e_j}(e_j\epsilon + \nu_j((fg)_{\delta(fg)})) \\
&= \delta(f) + \delta(g) - \frac{1}{e_j}(\nu_j((1)_\epsilon) + \nu_j((fg)_{\delta(fg)})) \\
&= \delta(f) + \delta(g) - \frac{1}{e_j}\nu_j((fg)_{\delta(fg)+\epsilon}) = \delta(f) + \delta(g) - \frac{1}{e_j}\nu_j((fg)_{\delta(f)+\delta(g)}) \\
&= \delta(f) + \delta(g) - \frac{1}{e_j}(\nu_j((f)_{\delta(f)}) + \nu_j((g)_{\delta(g)})) = \hat{\delta}_j(f) + \hat{\delta}_j(g)\,.
\end{aligned}$$

Additivity: Let $d := \delta(f) - \delta(g)$. Note that

$$\begin{aligned}
\hat{\delta}_j(f) \geq \hat{\delta}_j(g) &\iff \delta(f) - \frac{\nu_j((f)_{\delta(f)})}{e_j} \geq \delta(g) - \frac{\nu_j((g)_{\delta(g)})}{e_j} \\
&\iff \nu_j((f)_{\delta(f)}) \leq de_j + \nu_j((g)_{\delta(g)}).
\end{aligned} \tag{$*$}$$

Moreover, strict inequality in any one of the expressions of $(*)$ implies strict inequality in the other expressions of $(*)$. The remainder of the proof splits into several cases.

Case 1: $d > 0$. In this case $\delta(f+g) = \delta(f)$ and $(f+g)_{\delta(f+g)} = (f)_{\delta(f)} + ((1)_1)^d(g)_{\delta(g)}$. Then $\hat{\delta}_j(f+g) = \delta(f+g) - \frac{1}{e_j}\nu_j((f+g)_{\delta(f+g)}) = \delta(f) - \frac{1}{e_j}\nu_j((f)_{\delta(f)} + ((1)_1)^d(g)_{\delta(g)})$.

Subcase 1(a): $\hat{\delta}_j(f) > \hat{\delta}_j(g)$. In this case due to $(*)$ it follows that $\nu_j((f)_{\delta(f)}) < de_j + \nu_j((g)_{\delta(g)}) = \nu_j(((1)_1)^d) + \nu_j((g)_{\delta(g)}) = \nu_j(((1)_1)^d(g)_{\delta(g)})$. Therefore $\nu_j((f)_{\delta(f)} + ((1)_1)^d(g)_{\delta(g)}) = \nu_j((f)_{\delta(f)})$. Hence $\hat{\delta}_j(f+g) = \delta(f) - \frac{1}{e_j}\nu_j((f)_{\delta(f)}) = \hat{\delta}_j(f)$.

Subcase 1(b): $\hat{\delta}_j(f) = \hat{\delta}_j(g)$. As in 1(a) it follows by making use of $(*)$ that $\nu_j((f)_{\delta(f)}) = \nu_j(((1)_1)^d(g)_{\delta(g)})$, and therefore $\nu_j((f)_{\delta(f)} + ((1)_1)^d(g)_{\delta(g)}) \geq \nu_j((f)_{\delta(f)})$. Hence $\hat{\delta}_j(f+g) \leq \delta(f) - \frac{1}{e_j}\nu_j((f)_{\delta(f)}) = \hat{\delta}_j(f)$.

Case 2: $d = 0$. Let $e := \delta(f) = \delta(g)$ and $\epsilon := e - \delta(f+g) \geq 0$. It follows that $\hat{\delta}_j(f+g) = \delta(f+g) - \frac{1}{e_j}\nu_j((f+g)_{\delta(f+g)}) = \delta(f) - \epsilon - \frac{1}{e_j}\nu_j((f+g)_{\delta(f+g)}) = \delta(f) - \frac{1}{e_j}\nu_j(((1)_1)^\epsilon(f+g)_{\delta(f+g)}) = e - \frac{1}{e_j}\nu_j((f+g)_{\delta(f+g)+\epsilon}) = e - \frac{1}{e_j}\nu_j((f+g)_e) = e - \frac{1}{e_j}\nu_j((f)_e + (g)_e)$.

Subcase 2(a): $\hat{\delta}_j(f) > \hat{\delta}_j(g)$. In this case due to $(*)$ with $d = 0$ it follows that $\nu_j((f)_e) < \nu_j((g)_e)$. Therefore $\nu_j((f)_e + (g)_e) = \nu_j((f)_e)$. Hence $\hat{\delta}_j(f+g) = e - \frac{1}{e_j}\nu_j((f)_e) = \hat{\delta}_j(f)$.

Subcase 2(b): $\hat{\delta}_j(f) = \hat{\delta}_j(g)$. As in 2(a) $(*)$ implies that $\nu_j((f)_e) = \nu_j((g)_e)$. Consequently $\nu_j((f)_e + (g)_e) \geq \nu_j((f)_e)$. Hence $\hat{\delta}_j(f+g) \leq e - \frac{1}{e_j}\nu_j((f)_e) = \hat{\delta}_j(f)$.

Combining above conclusions it follows that $\hat{\delta}_j$ is indeed a semidegree. As we remarked earlier in the proof, conclusions of theorem 2.2.11 now follow from the uniqueness of the minimal presentation of a subdegree (proposition 2.2.3). □

Let X be an affine algebraic variety and δ be a finitely generated subdegree on $A := \mathbb{K}[X]$ with a minimal presentation $\delta = \max_{1 \leq i \leq N} \delta_i$. Fix i, $1 \leq i \leq N$. Theorem 2.2.11 implies that δ_i is integer-valued. Let d_i be the positive generator of the subgroup of $\mathbb{Z}$ generated by $\{\delta_i(f) : f \in A\}$. Then $\nu_i(\cdot) := -\frac{1}{d_i}\delta_i(\cdot)$ is a *discrete valuation* on

$\mathbb{K}(X)$. Recall (corollary 2.2.5) that δ_i corresponds to an irreducible component V_i of $X_\infty := X^\delta \setminus X$. We next show that the local ring $\mathcal{O}_{V_i,X^\delta}$ of X^δ at V_i is precisely the valuation ring of ν_i. In particular, $\mathcal{O}_{V_i,X^\delta}$ is *regular*, i.e. is a discrete valuation ring.

Proposition 2.2.12. *Let X, δ and A be as above. For each i, the local ring $\mathcal{O}_{V_i,X^\delta}$ is regular and hence is a discrete valuation ring. The valuation associated with $\mathcal{O}_{V_i,X^\delta}$ is precisely $\nu_i(\cdot) := -\frac{1}{d_i}\delta_i(\cdot)$.*

Proof. Fix an i, $1 \leq i \leq N$. Let $\mathfrak{p}_i$ be the prime ideal of A^δ corresponding to δ_i. Then $\mathcal{O}_{V_i,X^\delta}$ is the degree zero part of the local ring $A^\delta_{\mathfrak{p}_i}$. Let us identify A^δ with $\sum F_d t^d$. Then

$$\begin{aligned} \mathcal{O}_{V_i,X^\delta} &= \{\frac{ft^{kd}}{(gt^d)^k} : gt^d \notin \mathfrak{p}_i,\ d \geq \delta(g),\ k \geq 0\} \\ &= \{\frac{f}{g^k} : (g)_{\delta(g)} \notin \mathfrak{p}_i,\ \delta(f) \leq k\delta(g),\ k \geq 0\}. \end{aligned} \tag{2.8}$$

Let $R_i \subseteq \mathbb{K}(X)$ be the valuation ring of the discrete valuation ν_i. We have to show that $\mathcal{O}_{V_i,X^\delta} = R_i$. Recall that $R_i := \{g_1/g_2 : g_1, g_2 \in A,\ g_2 \neq 0,\ \nu_i(g_1/g_2) \geq 0\}$. Pick $g_1, g_2 \in A$ such that $g_1/g_2 \in R_i$. Then $\nu_i(g_1/g_2) = \nu_i(g_1) - \nu_i(g_2) \geq 0$ and hence $\delta_i(g_1) \leq \delta_i(g_2)$. Pick $f_i \in A$ such that $\delta_i(f_i) > \delta_j(f_i)$ for all $j \neq i$. It follows due to (2.5) that there is $k \geq 1$ such that $\delta(g_l f_i^k) = \delta_i(g_l f_i^k)$ for $l = 1, 2$. Then $\delta(g_1 f_i^k) = \delta_i(g_1 f_i^k) = \delta_i(g_1) + \delta_i(f_i^k) \leq \delta_i(g_2) + \delta_i(f_i^k) = \delta_i(g_2 f_i^k) = \delta(g_2 f_i^k)$. Moreover, lemma 2.2.1.2 implies that $(g_l f_i^k)_{\delta(g_l f_i^k)} \notin \mathfrak{p}_i$ for $l = 1, 2$. Then, according to (2.8), $g_1 f_i^k/(g_2 f_i^k) = g_1/g_2 \in \mathcal{O}_{V_i,X^\delta}$. Consequently $R_i \subseteq \mathcal{O}_{V_i,X^\delta}$. Therefore $\mathcal{O}_{V_i,X^\delta} = R_i$ due to

Lemma 2.2.12.1. *Let R be a discrete valuation ring and K be the quotient field of R. If S is a proper subring of K such that $R \subseteq S$, then $R = S$.*

Proof. Let ν be the discrete valuation associated to R and $h \in R$ be a parameter for ν, in particular $\nu(h) = 1$. Assume contrary to the claim that $R \neq S$. Let $f \in S \setminus R$. Then $f = u/h^k$ for some unit u of R and $k > 0$. It follows that $h^{-1} = u^{-1}fh^{k-1} \in S$. Let $g \in K \setminus \{0\}$. Then $\nu(gh^{-\nu(g)}) = 0$ and therefore $gh^{-\nu(g)} \in R \subseteq S$ and also $g =$

$gh^{-\nu(g)} \cdot h^{\nu(g)} \in S$. Therefore $S = K$, contrary to the assumptions, which completes the proof. □

To summarize, we have proved that $\mathcal{O}_{V_i,X^\delta}$ is precisely the valuation ring of ν_i. Since valuations are completely determined by their valuation rings [36, Section VI.8], it follows that ν_i is the valuation corresponding to $\mathcal{O}_{V_i,X^\delta}$, which completes the proof of the proposition. □

Example 2.2.13. Let $\mathcal{P}$ be a convex integral polytope of dimension n containing the origin in its interior. As in example 2.1.3, for each facet $\mathcal{Q}$ of $\mathcal{P}$, let $\omega_\mathcal{Q}$ be the smallest 'outward pointing' integral vector normal to $\mathcal{Q}$ and $c_\mathcal{Q} := \langle \omega_\mathcal{Q}, \alpha \rangle$, where α is any element of the hyperplane spanned by $\mathcal{Q}$. Recall that the toric completion $X_\mathcal{P}$ of $X := (\mathbb{K}^*)^n$ is isomorphic to X^δ for a subdegree δ on $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ and the minimal presentation of δ is:

$$\delta = \max_{\mathcal{Q} \text{ is a facet of } \mathcal{P}} \delta_\mathcal{Q},$$

where $\delta_\mathcal{Q}(x^\alpha) := k\frac{\langle \omega_\mathcal{Q}, \alpha \rangle}{c_\mathcal{Q}}$ as in example 2.1.3 (in particular k is such that $\delta_\mathcal{Q}$ takes integer values). Since the greatest common divisor of the components of $\omega_\mathcal{Q}$ is 1, it follows that the positive greatest common divisor of $\{\delta_\mathcal{Q}(f) : f \in A\}$ is $d_\mathcal{Q} := \frac{k}{c_\mathcal{Q}}$. Therefore, due to proposition 2.2.12, the order of zero of x^α along $V_\mathcal{Q}$ is $\operatorname{ord}_\mathcal{Q}(x^\alpha) = -\delta_\mathcal{Q}(x^\alpha)/d_\mathcal{Q} = -\langle \omega_\mathcal{Q}, \alpha \rangle$. This provides a proof of assertion 1 of theorem 2.0.5.

2.2.3 Finitely generated subdegrees with associated filtrations preserving intersections at ∞.

The main existence theorem below provides a link between the contents of this chapter and that of section 1.3. We start with a definition in the spirit of section 1.3.

Definition. Let δ and η be degree like functions on A.

- *η preserves the intersections at ∞ for the completion determined by δ* (in short, *for δ*), provided that for any closed subsets $X_1, \ldots, X_k$ of Spec A such that the closures of X_i in $\operatorname{Proj} A^\delta$ have an empty intersection at infinity, also the closures of X_i in $\operatorname{Proj} A^\eta$ have an empty intersection with $\operatorname{Proj} A^\eta \setminus \operatorname{Spec} A$.

- We say η is *integral* over δ provided $\eta(f) \leq \delta(f)$ for all $f \in A$, so that there is a natural inclusion $A^\delta \hookrightarrow A^\eta$ and A^η is integral over A^δ under the above inclusion.

Lemma 2.2.14. *Let δ and η be degree like functions on A. Assume there is a commuting diagram as follows:*

$$\begin{array}{ccc} & \operatorname{Spec} A & \\ \swarrow & & \searrow \\ \operatorname{Proj} A^\eta & \longrightarrow & \operatorname{Proj} A^\delta \end{array}$$

where the maps corresponding to the slanted arrows are the natural embeddings associated with δ and η. Then η preserves the intersections at ∞ for δ.

Proof. Straightforward from the definitions. □

Corollary 2.2.15. *Let δ and η be non-negative degree like functions on A. If η is integral over δ or $\eta = n\delta$ for some $n > 0$, then η preserves the intersections at ∞ for δ.*

Proof. If η is integral over δ, then the inclusion $A^\delta \subseteq A^\eta$ induces a morphism $\operatorname{Proj} A^\eta \to \operatorname{Proj} A^\delta$ (since the irrelevant ideal A^η_+ of A^η is integral over the irrelevant ideal A^δ_+ of A^δ) which is identity on $\operatorname{Spec} A$. Thus by lemma 2.2.14 η preserves the intersections at ∞ for δ.

Now assume $\eta = n\delta$. Let $\mathcal{F} := \{F_d\}_{d\geq 0}$ (resp. $\mathcal{G} := \{G_d\}_{d\geq 0}$) be the filtration corresponding to δ (resp. η). Recall that $A^\delta \cong \bigoplus_{d\geq 0} F_d t^d$ and $A^\eta \cong \bigoplus_{d\geq 0} G_d s^d$, where s, t are indeterminates over A. Note that for each $d \geq 0$, $G_d := \{f \in A : \eta(f) \leq d\} = \{f \in A : \delta(f) \leq d/n\} = F_{\lfloor d/n \rfloor}$. Therefore $A^\eta \cong \bigoplus_{d\geq 0} F_{\lfloor d/n \rfloor} s^d$. Sending $t \in A^\delta$ to $s^n \in A^\eta$ induces an inclusion of A^δ into A^η such that A^η is integral over A^δ. At this point the argument of the previous paragraph completes the proof of the corollary. □

Theorem 2.2.16 (Main Existence Theorem, see [25, Theroem 2.3] and [26, Theorem 2.2.9])**.** *Let δ be a finitely generated degree like function on A. Let I be the ideal generated by $(1)_1$ in A^δ. Then*

1. *there is a positive integer e and a finitely generated subdegree $\tilde{\delta}$ on A such that for all $h \in A$,*

$$\tilde{\delta}(h) := e \lim_{m\to\infty} \frac{\delta(h^m)}{m}.$$

2. *Subdegree $\tilde{\delta}$ is integral over $e\delta$ and ring $A^{\tilde{\delta}}$ is integral over A^{δ}. If δ is non-negative (resp. complete), then $\tilde{\delta}$ is also non-negative (resp. complete), and $\tilde{\delta}$ preserves the intersections at ∞ for δ.*

Proof. Fix $h \in A$ and $m \in \mathbb{N}$. Therefore $\delta(h^m) \le m\delta(h)$. Moreover, since ideal I is generated by $(1)_1$, it follows that $k := m\delta(h) - \delta(h^m)$ is the largest integer such that $(h^m)_{m\delta(h)} \in I^k$. The definition of ν_I implies that $\nu_I(((h)_{\delta(h)})^m) = k = m\delta(h) - \delta(h^m)$. Therefore $\delta(h^m)/m = \delta(h) - \nu_I(((h)_{\delta(h)})^m)/m$. It follows according to assertion 1 of Rees' theorem that $\bar{\delta}(h) := \lim_{m\to\infty} \delta(h^m)/m = \delta(h) - \bar{\nu}_I((h)_{\delta(h)})$ is well defined and there exists a positive integer e (independent of h) such that $\bar{\delta}(h) \in \frac{1}{e}\mathbb{Z}$, which proves the displayed formula of assertion 1.

To complete the proof of assertion 1 it remains to show that $\tilde{\delta}$ is a finitely generated subdegree. Let $\mathcal{F} := \{F_d\}_{d\in\mathbb{Z}}$ be the filtration on A corresponding to δ. As usual, we identify A^{δ} with $\bigoplus_{i\in\mathbb{Z}} F_i t^i$. For $m \in \mathbb{Z}$, let $\bar{F}_{\frac{m}{e}} := \{f \in A : \bar{\delta}(h) \le \frac{m}{e}\}$ and define $A^{\bar{\delta}} := \bigoplus_{m\in\mathbb{Z}} \bar{F}_{\frac{m}{e}} t^{\frac{m}{e}}$. Since $\bar{\delta} \le \delta$, it follows that $F_k \subseteq \bar{F}_k$ for each $k \in \mathbb{Z}$. Therefore $A^{\delta} \subseteq A^{\bar{\delta}}$. We will make use of the following

Claim 2.2.16.1. *$A^{\bar{\delta}}$ is integral over A^{δ}.*

Proof. It suffices to show that for each $h \in A$, $ht^{\bar{\delta}(h)}$ is integral over A^{δ}. Pick $h \in A$. Then $ht^{\bar{\delta}(h)}$ is integral over A^{δ} if and only if $\bar{H} := (ht^{\bar{\delta}(h)})^e$ is integral over A^{δ}. Note that $e\bar{\delta}(h) = \bar{\delta}(h^e)$ by construction of $\bar{\delta}$, and therefore $\bar{H} = h^e t^{\bar{\delta}(h^e)}$. Let $H := h^e t^{\delta(h^e)} \in A^{\delta}$ and $k := \bar{\nu}_I(H)$, where I is the ideal generated by $(1)_1$ in A^{δ}. Since $\nu_I(H^m) = m\delta(H) - \delta(H^m)$ and $\delta(H) = \delta(h^e)$, it follows that $k = \delta(h^e) - \bar{\delta}(h^e) = \delta(h^e) - e\bar{\delta}(h)$. Then k is an integer. Hence according to assertion 2 of Rees' theorem, H is in the integral closure of I^k in A^{δ}, i.e. H satisfies an equation of the form $H^l + G_1 H^{l-1} + \cdots + G_l = 0$, where $G_i \in I^{ik}$

for each i. Comparing the coefficients at $t^{l\delta(h^e)}$ in the above equation, we may assume w.l.o.g. that each G_i is of the form $g_i t^{i\delta(h^e)}$ for some $g_i \in A$. Since $G_i \in I^{ik}$, it follows that $i\delta(h^e) \geq \delta(g_i) + ik$, implying that $g_i t^{i(\delta(h^e)-k)}$ is an element of A^δ. Moreover, in the ring $A^{\bar{\delta}}$, (via embedding $A^\delta \hookrightarrow A^{\bar{\delta}}$) element $H = h^e t^{\delta(h^e)} = h^e t^{\bar{\delta}(h^e)+k} = h^e t^{\bar{\delta}(h^e)} t^k = t^k \bar{H}$. Substituting these values of H and G_i into the equation of integral dependence for H and then canceling a factor of t^{lk} we conclude that $(\bar{H})^l + \sum_{i=1}^{l} g_i t^{i(\delta(h^e)-k)} (\bar{H})^{l-i} = 0$. But then $\bar{H}$ is integral over A^δ, which completes the proof of the claim. □

Let $\tilde{\mathcal{F}} := \{\tilde{F}_d\}_{d\in\mathbb{Z}}$ be the filtration corresponding to $\tilde{\delta} = e\bar{\delta}$. Since $\tilde{\delta} \leq e\delta$, $A^{e\delta} \subseteq A^{\tilde{\delta}}$. But observe that $\tilde{F}_d = \{f : e\bar{\delta}(h) \leq d\} = \bar{F}_{\frac{d}{e}}$, and it follows that the homomorphism $\chi : A^{\bar{\delta}} := \bigoplus_{d\in\mathbb{Z}} \bar{F}_{\frac{d}{e}} t^{\frac{d}{e}} \longrightarrow \bigoplus_{d\in\mathbb{Z}} \tilde{F}_d s^d \cong A^{\tilde{\delta}}$ that sends $t \mapsto s^e$ and is the identity map on the coefficients (i.e. on $\bar{F}_{\frac{d}{e}}$ for $d \in \mathbb{Z}$) is an isomorphism of $\mathbb{K}$-algebras. Arguments similar to those in the conclusion of the proof of corollary 2.2.15 show that the restriction of χ to A^δ induces a chain of inclusions $A^\delta \subseteq A^{e\delta} \subseteq A^{\tilde{\delta}}$ and that $A^{e\delta}$ is integral over A^δ. Also, due to claim 2.2.16.1 it follows that $A^{\tilde{\delta}}$ is integral over A^δ, as required. Therefore $A^{\tilde{\delta}}$ is integral over $A^{e\delta}$, implying that $\tilde{\delta}$ is integral over $e\delta$. Moreover, since A^δ is a finitely generated $\mathbb{K}$-algebra, it follows that $A^{\tilde{\delta}}$ is a finitely generated $\mathbb{K}$-algebra.

From the constructive definition of $\tilde{\delta}$ it follows that $\tilde{\delta}(f^m) = m\tilde{\delta}(f)$ for all f and m. Then corollary 2.2.2 implies that $\tilde{\delta}$ is a subdegree. This completes the proof of the first assertion of the theorem.

If δ is non-negative (resp. complete), then by construction $\tilde{\delta}$ is also non-negative (resp. complete), and applying lemma 2.2.15 the second assertion follows as well, which completes the proof of the theorem. □

Remark 2.2.17. Let X be an affine variety with coordinate ring A and δ be a finitely generated degree like function on A. Form subdegree $\tilde{\delta}$ as in theorem 2.2.16. If in addition X is normal, then according to corollary 2.2.8 $X^{\tilde{\delta}}$ is also normal. Since $A^{\tilde{\delta}}$ is integral over A^δ (assertion 2 of theorem 2.2.16), it follows that there is a finite map $X^{\tilde{\delta}} \to X^\delta$

which is identity on X. But then $X^{\tilde{\delta}}$ must be the normalization of X^{δ}. Motivated by the latter, we will refer to $\tilde{\delta}$ as the *normalization* of δ.

Example 2.2.18. Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^n$ and $\mathcal{P}$ be the convex hull of $\mathcal{A}$ in $\mathbb{R}^n$. Assume that $\mathbb{Z}^n = \mathbb{Z}\mathcal{A}$ and that $\mathcal{P}$ contains the origin in its interior. Let $X_{\mathcal{A}}$ and $X_{\mathcal{P}}$ be as in section 2.0.1 (with arbitrary algebraically closed field $\mathbb{K}$ in place of $\mathbb{C}$). Recall from example 2.1.3 that $X_{\mathcal{P}}$ is isomorphic to the completion of $X := (\mathbb{K}^*)^n$ corresponding to a subdegree δ on $A := \mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ defined by:

$$\delta(\sum a_\alpha x^\alpha) := \max_{a_\alpha \neq 0} e\delta'(x^\alpha),$$

where $\delta'(x^\alpha) := \inf\{r \in \mathbb{Q}_+ : \alpha \in r\mathcal{P}\}$ and e is a suitable positive integer chosen to ensure that δ is integer valued. Let η be the degree like function on A corresponding to the completion $\phi_{\mathcal{A}} : (\mathbb{K}^*)^n \hookrightarrow X_{\mathcal{A}} \subseteq \mathbb{P}^{|\mathcal{A}|-1}(\mathbb{K})$ (recall that components of map $\phi_{\mathcal{A}}$ are the monomials x^α with $\alpha \in \mathcal{A}$). We claim that δ is the *normalization* of η. Once verified, remark 2.2.17 would imply that $X_{\mathcal{P}}$ is the normalization of $X_{\mathcal{A}}$ (cf. theorem 2.0.2).

Indeed, the definition of η implies that $\eta(x^\alpha) = \min\{k : \exists\ \alpha_1, \ldots, \alpha_k \in \mathcal{A}$ such that $\alpha = \sum_{i=1}^k \alpha_i\}$, and $\eta(\sum a_\alpha x^\alpha) = \max_{a_\alpha \neq 0} \eta(x^\alpha)$. Note that if $\eta(x^\alpha) \leq k$ then $\alpha \in k\mathcal{P}$. Hence $\delta \leq e\eta$. Since δ is a subdegree, it follows that $\delta \leq e\bar{\eta}$, where as in theorem 2.2.16, $\bar{\eta}(h) := \lim_{m\to\infty} \eta(h^m)/m$ for all $h \in A$. Let $\alpha \in \mathbb{Z}^n$ and $d := \delta(x^\alpha)$. Then $\alpha \in \frac{d}{e}\mathcal{P}$ and hence there is an expression of the form: $\alpha = \frac{d}{e}\sum r_i\alpha_i$ such that $\alpha_i \in \mathcal{A}$ for each i and r_i's are positive rational numbers such that $\sum_i r_i \leq 1$. Let $k \in \mathbb{N}$ be such that kdr_i is an integer for all i. Then $ke\alpha = \sum_i kdr_i\alpha_i$, and the definition of η implies that $\eta(x^{ke\alpha}) \leq \sum_i kdr_i = kd$. Therefore $\bar{\eta}(x^\alpha) \leq \frac{kd}{ke} = \frac{d}{e} = \frac{1}{e}\delta(x^\alpha)$, so that $\delta(x^\alpha) \geq e\bar{\eta}(x^\alpha)$. It follows that $\delta = e\bar{\eta}$ and δ is the normalization of η, as claimed.

We summarize the results of theorems 2.2.16, 1.3.2 and 1.3.7 in the following

Corollary 2.2.19. *Let X be an affine variety of dimension n and A be the coordinate ring of X.*

1. *Let $V_1, \ldots, V_m$ be closed subvarieties of X such that $\bigcap_i V_i$ is a finite set. Then there is a complete subdegree δ on A such that the corresponding completion ψ_δ of X preserves the intersection of the V_i's at ∞.*

2. *Let $f : X \to Y$ be a dominating map of affine varieties. Then there is a complete subdegree δ on A such that ψ_δ preserves map f at ∞.* □

Example 2.2.20. None of the assertions of corollary 2.2.19 would remain valid if we replace in its conclusion the 'subdegree' by a 'semidegree'. Let $X = Y = \mathbb{K}^2$ and $f := ((x_1^2 - x_2^4)^2 + x_1x_2, (x_1^2 - x_2^4)^3 + x_1x_2) : X \to Y$. Then f is a quasifinite map. We show below that there is *no* complete semidegree δ on $A := \mathbb{K}[x_1, x_2]$ such that ψ_δ preserves $\{f_1, f_2\}$ at ∞ over *any* point of Y.

Let δ be any complete semidegree on A with associated filtration $\mathcal{F} = \{F_d\}_{d\geq 0}$. Recall the notation of section 1.3: given an ideal $\mathfrak{a}$ of A, let $\mathfrak{a}^\delta := \bigoplus_{d\geq 0}(\mathfrak{a} \cap F_d)$ be the corresponding homogeneous ideal of A^δ. Let $a := (a_1, a_2) \in \mathbb{K}^2$ and $\mathfrak{p}_i = \langle f_i - a_i \rangle \subseteq A$ for $i = 1, 2$. If suffices to show that ψ_δ does not preserve $\{f_1, f_2\}$ at ∞ over a, which is equivalent to showing that $\sqrt{\mathcal{I}} \subsetneqq A^\delta_+$ (lemma 1.3.1), where $\mathcal{I} := \langle \mathfrak{p}_1^\delta, \mathfrak{p}_2^\delta, (1)_1 \rangle \subseteq A^\delta$ and A^δ_+ is the irrelevant ideal of A^δ. We will make use of the following two simple lemmas in the proof of the latter.

Lemma 2.2.21. *Let $\mathcal{I}$ and A^δ be as above. If $\mathcal{I}$ is generated by $k < 3$ elements, then $\sqrt{\mathcal{I}} \neq A^\delta_+$.*

Proof. Assume to the contrary that $\sqrt{\mathcal{I}} = A^\delta_+$. Since A^δ_+ is a maximal ideal of A^δ, it follows that $\mathcal{I}$ is an A^δ_+-*primary* ideal of A^δ [1, Proposition 4.2] and therefore dimension of $\operatorname{Spec} A^\delta$ is at most k ([1], Theorems 11.14 and 11.25). But dimension of $\operatorname{Spec} A^\delta$ is $1 + \dim(\operatorname{Spec} A) = 3 > k$ and the derived contradiction proves the lemma. □

Lemma 2.2.22. *Assume δ is a semidegree. Let $f \neq 0 \in A$ and $\mathfrak{a}$ be the ideal generated by f in A. Then ideal $\mathfrak{a}^\delta$ is generated by $(f)_{\delta(f)}$.*

Proof. If $g \in \mathfrak{a} \cap F_d$, then $g = fh$ for some $h \in A$ and $\delta(g) \leq d$. Since δ is a semidegree, $\delta(g) = \delta(f) + \delta(h)$ and hence $(g)_{\delta(g)} = (f)_{\delta(f)}(h)_{\delta(h)}$. But then $(g)_d = (g)_{\delta(g)}(1)_{d-\delta(g)} = (f)_{\delta(f)}(h)_{\delta(h)}(1)_{d-\delta(g)} \in \langle (f)_{\delta(f)} \rangle$. It follows that $\mathfrak{a}^\delta \subseteq \langle (f)_{\delta(f)} \rangle$, which completes the proof (since the inclusion in the opposite direction is obvious). □

We return to example 2.2.20. Let $d_i := \delta(x_i)$ for $i = 1, 2$. We show $\sqrt{\mathcal{I}} \subsetneqq A_+^\delta$, splitting the proof into several cases:

Case 1: $d_1 > 2d_2$. In this case $\delta(f_1 - a_1) = \delta(x_1^4) = 4d_1$, and $\delta(f_1 - a_1 - x_1^4) < 4d_1$. It follows that $(f_1 - a_1)_{4d_1} - (x_1^4)_{4d_1} \in \langle (1)_1 \rangle$. According to lemma 2.2.22, $\tilde{\mathfrak{p}}_1 = \langle (f_1 - a_1)_{4d_1} \rangle$ and therefore $\langle \tilde{\mathfrak{p}}_1, (1)_1 \rangle = \langle (f_1 - a_1)_{4d_1}, (1)_1 \rangle = \langle ((x_1)_{d_1})^4, (1)_1 \rangle$. Similarly $\langle \tilde{\mathfrak{p}}_2, (1)_1 \rangle = \langle ((x_1)_{d_1})^6, (1)_1 \rangle$. Hence $\mathcal{I} = \langle ((x_1)_{d_1})^4, (1)_1 \rangle$. Lemma 2.2.21 then implies that $\mathcal{I} \neq A_+^\delta$.

Case 2: $d_1 < 2d_2$. It follows by an argument similar to that of case 1 that $\mathcal{I} = \langle ((x_2)_{d_2})^8, (1)_1 \rangle$. Consequently an application of lemma 2.2.21 yields that $\sqrt{\mathcal{I}} \neq A_+^\delta$.

Case 3: $d_1 = 2d_2$. Let $d := d_1 = 2d_2$. Then $\delta(x_1 \pm x_2^2) \leq d$. On the other hand, since $\delta((x_1 - x_2^2) + (x_1 + x_2^2)) = d$, at least one of $\delta(x_1 + x_2^2)$ and $\delta(x_1 - x_2^2)$ is precisely d, whereas the other is ≥ 1 (since δ is complete). It follows that $d' := \delta(x_1^2 - x_2^4) = \delta(x_1 - x_2^2) + \delta(x_1 + x_2^2) \geq d + 1$. Therefore $\delta((x_1 - x_2^4)^2) = 2d' \geq 2d + 2 = 4d_2 + 2 > 3d_2 = \delta(x_1 x_2)$. The latter implies that $\delta(f_1 - a_1) = 2d'$ and also equality $(f_1 - a_1)_{2d'} = ((x_1^2 - x_2^4)^2)_{2d'} + (x_1 x_2)_{3d_2}((1)_1)^{2d' - 3d_2} - a_1((1)_1)^{2d'}$. Therefore $\langle \mathfrak{p}_1^\delta, (1)_1 \rangle = \langle ((x_1^2 - x_2^4)_{d'})^2, (1)_1 \rangle$. It similarly follows that $\langle \mathfrak{p}_2^\delta, (1)_1 \rangle = \langle ((x_1^2 - x_2^4)_{d'})^3, (1)_1 \rangle$. Consequently $\mathcal{I} = \langle ((x_1^2 - x_2^4)_{d'})^2, (1)_1 \rangle$ and lemma 2.2.21 implies that $\sqrt{\mathcal{I}} \neq A_+^\delta$.

Summarizing all cases considered, we have proved that $\sqrt{\mathcal{I}} \neq A_+^\delta$. Therefore ψ_δ does not preserve $\{f_1, f_2\}$ at ∞ over a.

Example 2.2.23. Let $f : X \to Y$ be as in example 2.2.20. For each complete semidegree δ on $A := \mathbb{K}[x_1, x_2]$, we have shown that ψ_δ does not preserve $\{f_1, f_2\}$ over any point in Y, and hence ψ_δ does not preserve map f at ∞. On the other hand, according to corollary 2.2.19 it follows that there exists a completion of X determined by a complete subdegree on A which preserves map f at ∞. Below we exhibit an example of such a subdegree, i.e. a complete subdegree which preserves map f at ∞.

We start with defining a complete filtration $\mathcal{F}$ on A by:

$$F_0 := \mathbb{K}\,, F_1 := \mathbb{K}\langle 1, x_2, x_1^2 - x_2^4\rangle\,, F_2 := (F_1)^2 + \mathbb{K}\langle x_1\rangle\,, F_d := \sum_{j=1}^{d-1} F_j F_{d-j} \text{ for } d \geq 3\,.$$

We claim $\delta := \delta_{\mathcal{F}}$ is a subdegree. Indeed, $A^\delta \cong \mathbb{K}[W, X_1, X_2, Z]/\langle W^3 Z - X_1^2 + X_2^4\rangle$ via the surjective $\mathbb{K}$-algebra homomorphism $\phi : \mathbb{K}[W, X_1, X_2, Z] \to A^\delta$ which sends $X_1 \mapsto (x_1)_2$, $X_2 \mapsto (x_2)_1$, $Z \mapsto (x_1^2 - x_2^4)_1$ and $W \mapsto (1)_1$. The pull back by ϕ of ideal I generated by $(1)_1$ in A^δ is $\langle W, X_1^2 - X_2^4\rangle$, which is a radical ideal. It follows that I itself is radical and therefore δ is a subdegree due to theorem 2.2.1. Moreover, one can show (by a lengthy calculation which we did not include) that $\delta = \max\{\delta_1, \delta_2\}$ where δ_1 is the weighted degree on A that assigns weight 1 to x_2 and -1 to $x_1 - x_2^2$, and δ_2 is the weighted degree on A that assigns weight 1 to x_2 and -1 to $x_1 + x_2^2$. Finally, the following claim completes the proof of the assertions of example 2.2.23.

Claim. *Completion ψ_δ associated to δ preserves map f at ∞.*

Proof. Let $a := (a_1, a_2) \in \mathbb{K}^2$ and $\xi := (\xi_1, \xi_2) : \mathbb{K}^2 \to \mathbb{K}^2$ be a nondegenerate linear transformation. Define $\mathfrak{p}_i := \langle (\xi_i \circ f)(x) - \xi_i(a)\rangle \subseteq \mathbb{K}[x, y]$ for $i = 1, 2$. Let $\mathcal{I} := \langle \mathfrak{p}_1^\delta, \mathfrak{p}_2^\delta, (1)_1\rangle \subseteq A^\delta$. It suffices to show that for a homogeneous prime ideal P of A^δ, if $P \supseteq \mathcal{I}$, then $P \supseteq \{(1)_1, (x_1)_2, (x_2)_1, (x_1^2 - x_2^4)_1\}$ (since the latter elements generate the irrelevant ideal A^δ_+).

Let $i = 1$ or 2. Let $\xi_i(x_1, x_2) := \xi_{i1}x_1 + \xi_{i2}x_2$. Then

$$(\xi_i \circ f)(x) = \xi_{i1}(x_1^2 - x_2^4)^2 + \xi_{i2}(x_1^2 - x_2^4)^3 + (\xi_{i1} + \xi_{i2})x_1x_2 \in A,$$

and therefore, in ring A^δ, elements $((\xi_i \circ f)(x) - \xi_i(a))_3 =$

$$\xi_{i1}((x_1^2 - x_2^4)_1)^2(1)_1 + \xi_{i2}((x_1^2 - x_2^4)_1)^3 + (\xi_{i1} + \xi_{i2})(x_1)_2(x_2)_1 - \xi_i(a)((1)_1)^3\,.$$

Let P be a homogeneous prime ideal of A^δ such that $P \supseteq \mathcal{I}$. Since $(1)_1 \in P$, it follows that $\xi_{i2}((x_1^2 - x_2^4)_1)^3 + (\xi_{i1} + \xi_{i2})(x_1)_2(x_2)_1 \in P$, $i = 1, 2$. The nondegeneracy of ξ implies that $(\xi_{i2}, \xi_{i1} + \xi_{i2})$, for $i = 1, 2$ are linearly independent. It follows that both $(x_1)_2(x_2)_1$ and $((x_1^2 - x_2^4)_1)^3$ are in P. Consequently $(x_1^2 - x_2^4)_1$ and either $(x_1)_2$ or $(x_2)_1$ are in P. Note that

$$((x_1)_2)^2 - ((x_2)_1)^4 = (x_1^2 - x_2^4)_4 = (x_1^2 - x_2^4)_1((1)_1)^3.$$

Since the element in the right hand side is in P, it follows that if either $(x_1)_2$ or $(x_2)_1$ is in P, then the other one is in P as well! Therefore we showed that $P \supseteq \{(x_1^2 - x_2^4)_1, (x_1)_2, (x_2)_1, (1)_1\}$, which completes the proof of the claim. $\square$

Chapter 3

Affine Bezout-type theorems

We introduce a family of *iterated* semidegrees extending the notion of weighted homogeneous degrees and prove an explicit Bezout-type bound on the number of solutions of a system of n polynomials on $\mathbb{K}^n$, which is sharp whenever the associated projective completion preserves the intersection of the component hypersurfaces at ∞. We also establish a Bezout-type bound in the general case of a subdegree.

3.0 Background

3.0.1 Valuations and convex bodies

We start with a brief description of the theory of convex bodies associated to a valuation on a variety and their relation to intersection theory on the variety. Our general geometric bound for semidegrees (unlike the explicit bound in the case of iterated semidegrees) makes use of the theorem we quote below from this theory. We follow the exposition of [17], where the theory was developed.

Let X be a quasi-projective variety of dimension n over an algebraically closed field $\mathbb{K}$. A *valuation* on the field $\mathbb{K}(X)$ of rational functions on X with value group $\mathbb{Z}^n$ (equipped with an addition preserving total order $\prec$) is a surjective map $\nu : \mathbb{K}(X) \setminus \{0\} \to \mathbb{Z}^n$ such that:

1. $\nu(\lambda f) = \nu(f)$ for all $\lambda \neq 0 \in \mathbb{K}$,
2. $\nu(f+g) \geq \max\{\nu(f), \nu(g)\}$ for all $f, g \neq 0 \in \mathbb{K}(X)$,
3. for every pair of elements $f, g \neq 0 \in \mathbb{K}(X)$ such that $\nu(f) = \nu(g)$, there is $\lambda \neq 0 \in \mathbb{K}$ such that $\nu(f - \lambda g) \succ \nu(f)$, and
4. $\nu(fg) = \nu(f) + \nu(g)$ for all $f, g \neq 0 \in \mathbb{K}(X)$.

Example 3.0.1 (*Monomial Valuation*)**.** Let x be any smooth point of X. Choosing a set of local parameters at x, one can embed $\mathbb{K}[X]$ into the completion $\hat{\mathcal{O}}_{x,X}$ of the local ring $\mathcal{O}_{x,X}$ of X at x. According to Cohen structure theorem [8, Theorem 7.7] $\hat{\mathcal{O}}_{x,X}$ is isomorphic to the ring $\mathbb{K}[[t_1, \ldots, t_n]]$ of formal power series in n indeterminates $t_1, \ldots, t_n$. Choose any addition preserving *well ordering* $\prec$ on $(\mathbb{Z}_+)^n$ (e.g. the 'lexicographic ordering' of $\mathbb{Z}^n$). Define $\nu(\sum a_\alpha t^\alpha) := \min_\prec\{\alpha : a_\alpha \neq 0\}$.Then map ν extends uniquely to the field of fractions of $\mathbb{K}[[t_1, \ldots, t_n]]$, and the restriction of ν on $\mathbb{K}(X)$ is a valuation in the above sense.

The *intersection index* $[L_1, \ldots, L_n]$ of finite dimensional vector spaces $L_1, \ldots, L_n$ of rational functions on X is the number of 'solutions' of the system $f_1 = f_2 = \cdots = f_n = 0$ for generic $f_i \in L_i$, where we count solutions only on the maximal Zariski open subset U of X with all functions in $\bigcup_{i=1}^n L_i$ being regular.

Let ν be a (surjective) valuation on X with values in $\mathbb{Z}^n$. Then every finite dimensional vector subspace L of rational functions on X admits an associated semigroup $S(L) := \{(k, \nu(f)) : f \in L^k,\ k \geq 0\} \subseteq \mathbb{N} \bigoplus \mathbb{Z}^n$. Let $C(L)$ be the closure of the convex hull of $S(L)$ in $\mathbb{R}^{n+1}$. Then $C(L)$ is a convex cone. Let $\Delta(L) := C(L) \cap (\{1\} \times \mathbb{R}^n)$. The main theorem of [17] gives a connection between $\Delta(L)$ and the *self intersection number* $[L, \ldots, L]$ of L.

Theorem 3.0.2 (see [17, Main theorem])**.** *Let L and ν be as above. Choose any basis $f_0, \ldots, f_l$ of L. Let Y_L be the the closure in $\mathbb{P}^l(\mathbb{K})$ of the image of the rational map $\Phi_L : X \to \mathbb{P}^l(\mathbb{K})$ defined by $f_0, \ldots, f_l$.*

1. *$\Delta(L)$ is a compact convex body and the real dimension of $\Delta(L)$ is equal to the dimension of Y_L as an algebraic variety over $\mathbb{K}$.*
2. *Assume $\dim_{\mathbb{K}}(Y_L) = n$. Then $[L, \dots, L] = \frac{n!d(L)}{s(L)} \operatorname{Vol}_n(\Delta(L))$, where Vol_n is the n-dimensional Euclidean volume, $d(L)$ is the mapping degree of Φ_L and $s(L)$ is the index in $\mathbb{Z}^n$ of the subgroup generated by all of the differences $\alpha - \beta$ such that $(k, \alpha), (k, \beta) \in S(L)$ for some $k \geq 1$.*

Remark. Although in [17] only the case $\mathbb{K} = \mathbb{C}$ is considered, the arguments used there to prove theorem 3.0.2 remain valid for all algebraically closed fields.

3.1 Bezout Theorem for Semidegrees

Let X be an n dimensional affine variety. Let δ be a complete degree like function on the coordinate ring A of X with associated filtration $\mathcal{F} := \{F_d : d \geq 0\}$. Recall from example 1.0.6 that there exists $d > 0$ such that $(A^\delta)^{[d]} := \bigoplus_{k \geq 0} F_{kd}$ is generated by F_d as a $\mathbb{K}$-algebra and then the d-uple embedding embeds X^δ into $\mathbb{P}^l(\mathbb{K})$, where $l := \dim_{\mathbb{K}} F_d - 1$.

Below we will make use of a notion of multiplicity at an isolated point b of fiber $f^{-1}(a)$ of morphism $f : X \to \mathbb{K}^n$, where X is an affine variety and $a := (a_1, \dots, a_n) \in \mathbb{K}^n$. The latter multiplicity we define following the definition in [10, Example 12.4.8] as the intersection multiplicity at b of the effective Cartier divisors determined by regular (on X) functions $f_j - a_j$, $1 \leq j \leq n$.

Theorem 3.1.1 (see [25, Theroem 1.3] and [26, Theorem 3.1.1])**.** *Let X, A, δ, d and l be as above. Denote by D the degree of X^δ in $\mathbb{P}^l(\mathbb{K})$. Let $f = (f_1, \dots, f_n) : X \to \mathbb{K}^n$ be any generically finite map. Then for all $a \in \mathbb{K}^n$,*

$$|f^{-1}(a)| \leq \frac{D}{d^n} \prod_{i=1}^{n} \delta(f_i) \tag{A}$$

where $|f^{-1}(a)|$ is the number of the isolated points in fiber $f^{-1}(a)$ each counted with the multiplicity of $f^{-1}(a)$ at the respective point. If in addition δ is a semidegree and ψ_δ

preserves $\{f_1, \ldots, f_n\}$ *at* ∞ *over* a*, then* (A) *holds with an equality.*

Proof. Let $a = (a_1, \ldots, a_n) \in \mathbb{K}^n$. For each i, let $\mathfrak{a}_i$ be the ideal generated by $(f_i - a_i)^d$ in A and $\mathfrak{a}_i^\delta := \bigoplus_{j \geq 0}(\langle (f_i - a_i)^d \rangle \cap F_j)$ be the corresponding *homogeneous* ideal in A^δ. Clearly $G_i := ((f_i - a_i)^d)_{dd_i} \in \mathfrak{a}_i$, where $d_i := \delta(f_i)$ for each $i = 1, \ldots, n$. It follows that

$$\bigcap_{i=1}^{n} \{x \in X : (f_i(x) - a_i)^d = 0\} \subseteq \bigcap_{i=1}^{n} \{x \in X^\delta : G(x) = 0 \text{ for all } G \in \mathfrak{a}_i^\delta\} \tag{3.1}$$

$$\subseteq \bigcap_{i=1}^{n} \{x \in X^\delta : G_i(x) = 0\} \tag{3.2}$$

where the first inclusion is due to lemma 1.3.1, since the closure of the hypersurface $V(\mathfrak{a}_i)$ of X in X^δ is $V(\mathfrak{a}_i^\delta)$. Note that the sum of the multiplicities of the intersections of Cartier divisors determined by $(f_i - a_i)^d$ at the isolated points in the set on the left hand side of (3.1) is precisely d^n times the sum of the multiplicities of fiber $f^{-1}(a)$ at the isolated points in $f^{-1}(a)$.

Pick a set of homogeneous coordinates $[y_0 : \cdots : y_L]$ of $\mathbb{P}^l(\mathbb{K})$. The d-uple embedding of X^δ into $\mathbb{P}^l(\mathbb{K})$ induces a surjective homomorphism $\phi : \mathbb{K}[y_0, \ldots, y_L] \to (A^\delta)^{[d]}$ of graded $\mathbb{K}$-algebras. For each i, choose an arbitrary homogeneous polynomial $\hat{G}_i \in \mathbb{K}[y_0, \ldots, y_L]$ of degree d_i such that $\phi(\hat{G}_i) = G_i$. According to the classical Bezout theorem on $\mathbb{P}^l(\mathbb{K})$, the sum of the multiplicities of isolated points of $X^\delta \cap V(\hat{G}_1) \cap \cdots \cap V(\hat{G}_n)$ in $\mathbb{P}^l(\mathbb{K})$ is at most $Dd_1 \cdots d_n$, and it is equal to $Dd_1 \cdots d_n$ if all the points in the intersection are isolated.

Since $X^\delta \cap V(\hat{G}_1) \cap \cdots \cap V(\hat{G}_n)$ is precisely $X^\delta \cap V(G_1) \cap \cdots \cap V(G_n)$, inequality (A) follows from (3.2) and the conclusions of the two preceding paragraphs. The last assertion of theorem 3.1.1 follows from the following observations:

1. if δ is a semidegree, then according to lemma 2.2.22 ideal $\mathfrak{a}_i^\delta$ is generated by G_i and hence $\subseteq$ in (3.2) can be replaced by $=$, and
2. completion ψ_δ preserves $\{f_1, \ldots, f_n\}$ at ∞ over a iff the $\subseteq$ in (3.1) is in fact an equality. □

Remark 3.1.2. Both assertions of theorem 3.1.1 are valid for $\delta = \max_{j=1}^{N} \delta_j$ being a subdegree with $\delta_1(f_i) = \cdots = \delta_N(f_i)$ for all $i = 1, \ldots, n$.

Remark 3.1.3. Let $f := (f_1, \ldots, f_n) : X \to \mathbb{K}^n$ be a quasifinite map and δ be a complete semidegree on $A := \mathbb{K}[X]$. We claim that if ψ_δ preserves $\{f_1, \ldots, f_n\}$ at ∞ over any point, then ψ_δ preserves $\{f_1, \ldots, f_n\}$ at ∞ over *all* points. Indeed, let $a := (a_1, \ldots, a_n) \in \mathbb{K}^n$. For each i, let $\mathfrak{p}_i(a)$ be the ideal generated by $(f_i - a_i)$ in A, and let $\mathfrak{p}_i^\delta(a) := \bigoplus_{j \geq 0}(\mathfrak{p}_i(a) \cap F_j) \subseteq A^\delta$. Due to lemma 1.3.1, ψ_δ preserves $\{f_1, \ldots, f_n\}$ at ∞ over a iff $\sqrt{\mathcal{I}(a)} = A_+^\delta$, where $\mathcal{I}(a) := \langle \mathfrak{p}_1^\delta(a), \ldots, \mathfrak{p}_n^\delta(a), (1)_1 \rangle \subseteq A^\delta$. According to lemma 2.2.22, $\mathfrak{p}_i^\delta(a) = \langle (f_i - a_i)_{d_i} \rangle$, where $d_i := \delta(f_i)$ for each i. Then $\mathcal{I}(a) = \langle (f_1 - a_1)_{d_1}, \ldots, (f_n - a_n)_{d_n}, (1)_1 \rangle = \langle (f_1)_{d_1}, \ldots, (f_n)_{d_n}, (1)_1 \rangle$. Since the latter expression is independent of a_i's, the claim follows. Note that we have shown (in the notation of remark 1.3.6) that either $S_{\psi_\delta} = \emptyset$ or $S_{\psi_\delta} = f(X)$.

Example 3.1.4 (Weighted Bezout theorem, cf. [7], [25, Example 7], [26, Example 3.1.4])**.** Let $X := \mathbb{K}^n$ and δ be a weighted degree on $A := \mathbb{K}[x_1, \ldots, x_n]$ which assigns weights $d_i > 0$ to x_i, $1 \leq i \leq n$. Then $(1)_1, (x_1)_{d_1}, \ldots, (x_n)_{d_n}$ is a set of $\mathbb{K}$-algebra generators of A^δ. A straightforward application of lemma 1.3.1 implies that ψ_δ preserves at ∞ all components of the identity map $\mathbb{1}$ of $\mathbb{K}^n$ over 0. Therefore theorem 3.1.1 with $f = \mathbb{1}$ and d as in the preamble to theorem 3.1.1 implies that $1 = \frac{D}{d^n} \prod_{i=1}^n d_i$ and therefore $D = \frac{d^n}{\prod_{i=1}^n d_i}$. Consequently, theorem 3.1.1 implies for any f that:

$$|f^{-1}(a)| \leq \prod_{i=1}^{n} \frac{\delta(f_i)}{d_i} \ . \tag{3.3}$$

Recall (example 2.1.1) that $\operatorname{gr} A^\delta \cong \mathbb{K}[x_1, \ldots, x_n]$ via an identification of $[(g)_{\delta(g)}] \in \operatorname{gr} A^\delta$ and $\mathfrak{L}_\delta(g) \in \mathbb{K}[x_1, \ldots, x_n]$, for $0 \neq g \in A$. Let $a := (a_1, \ldots, a_n) \in \mathbb{K}^n$ and $\mathfrak{a}_i := \langle f_i - a_i \rangle$ (cf. the proof of theorem 3.1.1). According to lemma 2.2.22, ideals $\mathfrak{a}_i^\delta$ are generated by

$(f_i - a_i)_{\delta(f_i)} = (f_i)_{\delta(f_i)} - a_i((1)_1)^{\delta(f_i)}$ for $1 \leq i \leq n$. Therefore,

$$\begin{aligned}
\text{(3.3) holds with equality} &\iff \psi_\delta \text{ preserves } \{f_1, \ldots, f_n\} \text{ at } \infty \text{ over } a \\
&\iff V(\mathfrak{a}_1^\delta, \ldots, \mathfrak{a}_n^\delta, (1)_1) = \emptyset \subseteq \operatorname{Proj} A^\delta \\
&\iff V((f_1)_{\delta(f_1)}, \ldots, (f_n)_{\delta(f_n)}, (1)_1) = \emptyset \subseteq \operatorname{Proj} A^\delta \\
&\iff V([(f_1)_{\delta(f_1)}], \ldots, [(f_n)_{\delta(f_n)}]) = \emptyset \subseteq \operatorname{Proj} \operatorname{gr} A^\delta \\
&\iff V(\mathfrak{L}_\delta(f_1), \ldots, \mathfrak{L}_\delta(f_n)) = \{0\} \in \mathbb{K}^n,
\end{aligned}$$

where the last relation holds since the maximal ideal of the origin in $\mathbb{K}^n$ corresponds to the irrelevant ideal $\bigoplus_{k>0} F_k/F_{k-1}$ of $\operatorname{gr} A^\delta$ via the isomorphism $\operatorname{gr} A^\delta \cong \mathbb{K}[x_1, \ldots, x_n]$. Finally note that formula (3.3) in combination with condition $V(\mathfrak{L}_\delta(f_1), \ldots, \mathfrak{L}_\delta(f_n)) = \{0\} \in \mathbb{K}^n$ for equality in (3.3) constitute the content of the *Weighted Bezout theorem* stated in section 0.2.

Applying theorem 3.0.2, number D of theorem 3.1.1 admits a geometric description as the volume of a convex body associated to δ and a $\mathbb{Z}^n$ valued (surjective) valuation ν of $\mathbb{K}(X)$ (cf. section 3.0), namely:

Proposition 3.1.5 (see [25, Proposition 1.4] and [26, Theorem 3.1.3])**.** *Let X, A, d, δ, l and D be as in theorem 3.1.1 and ν be a valuation on A with values in $\mathbb{Z}^n$. Let C be the smallest closed cone in $\mathbb{R}^{n+1}$ containing*

$$G := \{(\frac{1}{d}\delta(f), \nu(f)) : f \in A\} \cup \{(1, 0, \ldots, 0)\}.$$

Let Δ be the convex hull of the cross-section of C with the first coordinate having value 1. Then $D = n! \operatorname{Vol}_n(\Delta)$, where Vol_n is the n-dimensional Euclidean volume.

Proof. Consider $L := F_d = \{f \in A : \delta(f) \leq d\}$. As in section 3.0, we introduce $S(L) := \{(k, \nu(f)) : f \in L^k,\ k \geq 0\} \subseteq \mathbb{N} \bigoplus \mathbb{Z}^n$. Let $C(L)$ be the closure of the convex hull of $S(L)$ in $\mathbb{R}^{n+1}$ and $\Delta(L) := C(L) \cap (\{1\} \times \mathbb{R}^n)$. Mapping Φ_L of theorem 3.0.2 is precisely the embedding $X \subseteq X^\delta \hookrightarrow \mathbb{P}^l(\mathbb{K})$, so that the mapping degree $d(L)$ of Φ_L

is 1. Therefore theorem 3.0.2 implies that $[L, \ldots, L] = \frac{n!}{s(L)} \operatorname{Vol}_n(\Delta(L))$, where $s(L)$ is the index in $\mathbb{Z}^n$ of the subgroup $S'(L)$ generated by all the differences $\alpha - \beta$ such that $(k, \alpha), (k, \beta) \in S(L)$ for some $k \geq 1$. Note also that:

1. according to its definition $[L, \ldots, L]$ is equal to the degree D of X^δ in $\mathbb{P}^L(\mathbb{C})$,
2. F_d generates $(A^\delta)^{[d]}$, so that for $L^k = (F_d)^k = F_{kd}$ for all $k \geq 1$, and therefore for all $f \in A$, $(k, \nu(f)) \in S(L)$ for all $k \geq \frac{\delta(f)}{d}$, and
3. for all $k \geq 1$, $1 \in L^k$ and therefore $(k, \nu(1)) = (k, 0, \ldots, 0) \in S(L)$.

Properties 2 and 3 imply that $S'(L) \supseteq \{\nu(f) : f \in A\}$. Since ν is surjective, it follows that $S'(L) = \mathbb{Z}^n$ and hence $s(L) = 1$. But then $D = n! \operatorname{Vol}_n(\Delta(L))$ due to property 1. Finally, properties 2 and 3 also imply that $C = C(L)$, and consequently that $\Delta = \Delta(L)$. $\square$

Example 3.1.6. Let $X = \mathbb{K}^n$ and δ be as in example 3.1.4. Let ν be any monomial valuation on $A := \mathbb{K}[x_1, \ldots, x_n]$, e.g. the one that assigns to $\sum a_\alpha x^\alpha \in A \setminus \{0\}$ the lexicographically (coordinatewise) minimal exponent α among all $\alpha := (\alpha_1, \ldots, \alpha_n)$ with $a_\alpha \neq 0$. With d being any common multiple of $d_1, \ldots, d_n$, it is straightforward to see that $\Delta = \{(1, x) \in \mathbb{R}_+^{n+1} : \sum_{i=1}^n x_i d_i \leq d\}$ and therefore $\operatorname{Vol}_n(\Delta) = \frac{1}{n!} \prod_{i=1}^n \frac{d}{d_i}$. It follows that $D = n! \operatorname{Vol}_n(\Delta) = \prod_{i=1}^n \frac{d}{d_i}$, which is, of course, the value we have calculated in example 3.1.4.

3.2 Iterated Semidegree

In this section we describe particularly simple semidegrees generalizing weighted homogeneous degrees for which we establish a constructive version of affine Bezout-type theorem. Our dream is that a stronger version of Main Existence Theorem 2.2.16 would be valid with subdegrees in whose minimal presentations only constructive semidegrees 'like' the iterated semidegrees of this section would appear, and we expect that a precise constructive version of an affine Bezout-type theorem for any generically finite map $f : \mathbb{K}^n \to \mathbb{K}^n$ would follow.

Let A be a domain and δ be a degree like function on A. Pick $f \in A$ and an integer w with $w < \delta(f)$. Let s be an indeterminate over A and δ_e be a 'natural' extension of δ to $A[s]$ such that $\delta_e(s) = w$, namely: $\delta_e(\sum a_i s^i) := \max_{a_i \neq 0}(\delta(a_i) + iw)$. Of course δ is a degree like function iff δ_e is a degree like function.

Lemma 3.2.1. *δ is a semidegree iff δ_e is a semidegree.*

Proof. The ($\Leftarrow$) direction is obvious, since $\delta \equiv \delta_e|_A$. For the proof of the 'only if' implication, let $G := \sum g_i s^i, H := \sum h_j s^j \in A[s]$ with $d := \delta_e(G), e := \delta_e(H)$. For each $k \geq 0$, let $G_k := \sum_{\delta(g_i)+iw=k} g_i s^i$ and $H_k := \sum_{\delta(h_i)+iw=k} h_i s^i$. It suffices to show that $\delta_e(G_d H_e) = d + e$, and hence we may w.l.o.g. assume $G = G_d$ and $H = H_e$. Let i_0 (resp. j_0) be the largest integer such that $g_{i_0} \neq 0$ (resp. $h_{j_0} \neq 0$). Then $GH = g_{i_0} h_{j_0} s^{i_0+j_0} + \sum_{m<i_0+j_0} a_m s^m$. Thus $\delta_e(GH) \geq \delta_e(g_{i_0} h_{j_0} s^{i_0+j_0}) = \delta(g_{i_0} h_{j_0}) + (i_0 + j_0)w = \delta(g_{i_0}) + i_0 w + \delta(h_{j_0}) + j_0 w = d + e$. Since the inequality $\delta_e(GH) \leq d + e$ is obviously true, it follows that $\delta_e(GH) = d + e$. □

Remark 3.2.2. In fact δ is a subdegree iff δ_e is a subdegree, which follows by means of calculations similar to those in the proof of lemma 3.2.1 and of the characterization of subdegrees as degree like functions η satisfying $\eta(f^k) = k\eta(f)$ for all $f \in A$ and $k \geq 0$ (corollary 2.2.2).

Let J denote the ideal generated by $s - f$ in $A[s]$. Identify A with $A[s]/J$ and define $\tilde{\delta}$ to be the degree like function on A *induced by* δ_e, i.e. $\tilde{\delta}(g) := \min\{\delta_e(G) : G - g \in J\}$. Let $\mathfrak{a}$ be the principal ideal generated by f in A and let $\mathfrak{a}^\delta$ be the ideal induced in A^δ by $\mathfrak{a}$ (as defined in lemma 1.3.1). Denote by $\operatorname{gr}\mathfrak{a}$ the ideal generated by the image of $\mathfrak{a}^\delta$ in $\operatorname{gr} A^\delta$, i.e. $\operatorname{gr}\mathfrak{a} := \langle [(g)_{\delta(g)}] : g \in \mathfrak{a} \rangle$, where $[(g)_{\delta(g)}]$ denotes the equivalence class of $(g)_{\delta(g)}$ in $\operatorname{gr} A^\delta$.

Remark 3.2.3. A straightforward application of definitions shows that $\tilde{\delta}$ is a degree like function provided that δ_e is a degree like function and $\tilde{\delta} \equiv 0$ on $\mathbb{K}$. Note that $\tilde{\delta}$ is a meaningful degree like function only if $[(f)_{\delta(f)}]$ is not a unit in $\operatorname{gr} A^\delta$. Indeed, if $[(f)_{\delta(f)}]$ is a unit in $\operatorname{gr} A^\delta$, then $[(1)_0] = [(f)_d][(g)_{-d}] \in \operatorname{gr} A^\delta$ for some $g \in A$ with $\delta(g) = -d$. It follows that if $1 - fg \neq 0$ then $\delta(1 - fg) < 0$. Let $G := 1 - fg + gs \in A[s]$. Then $\delta_e(gs) = w - d < 0$ and therefore $\delta_e(G) < 0$. Moreover, $1 \equiv G \mod J$ in $A[s]$. Consequently $\tilde{\delta}(1) \leq \delta_e(G) < 0$. Since, for all $h \in A$ and $n \in \mathbb{Z}_+$, $\tilde{\delta}(h) = \tilde{\delta}(h \cdot (1)^n) \leq \tilde{\delta}(h) + n\tilde{\delta}(1)$, it follows that $\tilde{\delta}(h) = -\infty$ for all $h \in A$. Moreover, $(1)_1$ is a unit in $A^{\tilde{\delta}}$ (since $((1)_1)^{-1} = (1)_{-1}$) and therefore $\operatorname{gr} A^{\tilde{\delta}} \cong A^{\tilde{\delta}}/\langle (1)_1 \rangle$ is the zero ring.

Theorem 3.2.4 (cf. [25, Example 5] and [26, Theorem 2.1.3])**.**

1. *(a) If δ is non-negative and $w > 0$, then $\tilde{\delta}$ is non-negative.*
   (b) $\tilde{\delta}(g) \leq \delta(g)$ for all $g \in A$. If $[(g)_{\delta(g)}] \notin \operatorname{gr} \mathfrak{a}$, then $\tilde{\delta}(g) = \delta(g)$.
   (c) $\tilde{\delta}(f) \leq w < \delta(f)$. If δ is a semidegree and $[(f)_{\delta(f)}]$ is not a unit in $\operatorname{gr} A^\delta$, then $\tilde{\delta}(f) = w$.
2. *Assume δ is a semidegree. Then*
   (a) The homomorphism of $\mathbb{K}$-algebras $A[s]^{\delta_e} \cong A^\delta[s] \to A^{\tilde{\delta}}$ induced by the inclusion $A^\delta \hookrightarrow A^{\tilde{\delta}}$ (available due to $\tilde{\delta} \leq \delta$) and $s \mapsto (f)_w$ is surjective with kernel $\langle (s - f)_{\delta_e(s-f)} \rangle$.
   (b) $\operatorname{gr} A^{\tilde{\delta}}$ is isomorphic to $(\operatorname{gr} A^\delta / \operatorname{gr} a)[z]$, where z is an indeterminate of degree w and the isomorphism maps z to the class of equivalence $[(f)_w]$ of $(f)_w \in A^{\tilde{\delta}}$.
   (c) $\tilde{\delta}$ is a semidegree if and only if $\operatorname{gr} \mathfrak{a}$ is a prime ideal of $\operatorname{gr} A^\delta$.

Note that if δ is a semidegree, then $\operatorname{gr} \mathfrak{a}$ is a principal ideal in $\operatorname{gr} A^\delta$ generated by $[(f)_{\delta(f)}]$ due to lemma 2.2.22.

Proof. Assertion 1(a) is a straightforward consequence of the definitions. Note that $\tilde{\delta}(g) \leq \delta_e(G)$ for all $g \in A$ and $G \in A[s]$ such that $G \equiv g \mod J$. The first assertion of 1(b) follows by setting $G := g$ in the previous sentence. Similarly, the first assertion of 1(c) follows by setting $g := f$ and $G := s$. As for the second assertion of 1(b), let $g \in A$ and $G \in A[s]$ be such that $G \equiv g \mod J$. Let $a_0, \ldots, a_k \in A$ such that

$G = g + (s-f)(a_0 + a_1 s + \cdots + a_k s^k) = (g - fa_0) + \sum_{i=1}^{k}(a_{i-1} - fa_i)s^i + a_k s^{k+1}$. Note that if $e := \delta(g - fa_0) < d := \delta(g)$, then $\delta(fa_0) = d$ and $(g)_d = (fa_0)_d + ((1)_1)^{d-e}(g - fa_0)_e$, so that $[(g)_d] = [(fa_0)_d] \in \operatorname{gr}\mathfrak{a}$. In other words, if $[(g)_{\delta(g)}] \notin \operatorname{gr}\mathfrak{a}$, then $\delta(g - fa_0) \geq \delta(g)$ and therefore $\delta_e(G) \geq \delta(g)$. It follows that $\tilde{\delta}(g) \geq \delta(g)$, which concludes the proof of 1(b).

Next we prove the second assertion of 1(c). Assume δ is a semidegree and contrary to the conclusion of 1(c) that $\tilde{\delta}(f) < w$. Then it suffices to show that $[(f)_d]$ is a unit in $\operatorname{gr} A^\delta$, where $d := \delta(f)$. Indeed, $\tilde{\delta}(f) < w$ in view of the definition of $\tilde{\delta}$ in terms of δ_e implies that there is an identity

$$f = a_k f^k + a_{k+1} f^{k+1} + \cdots + a_l f^l \tag{3.4}$$

with $a_k, \ldots, a_l \in A$ such that for all j, $0 \leq k \leq j \leq l$, $\delta(a_j) + jw < w$. In particular, $\delta(a_0) < w$ if $a_0 \neq 0$ and $\delta(a_1) < 0$ if $a_1 \neq 0$.

If $k > 1$, then dividing both sides of (3.4) by f it follows that $fg = 1$, where $g := \sum_{j=k}^{l} a_j f^{j-2}$. Then $\delta(g) = -\delta(f) = -d$ and therefore $[(f)_d] \cdot [(g)_{-d}] = [(1)_0] \in \operatorname{gr} A^\delta$. Since $[(1)_0]$ is the identity in $\operatorname{gr} A^\delta$, it follows that $[(f)_d]$ is a unit in $\operatorname{gr} A^\delta$, which proves assertion 1(c) in the case that $k > 1$.

If $k = 1$, then (3.4) implies that $1 - a_1 = fg_1$, where $g_1 := a_2 + a_3 f + \cdots + a_l f^{l-2}$. Since $\delta(a_1) < 0$, it follows that $\delta(1 - a_1) = 0$ and therefore $\delta(g_1) = -d$. Moreover, $[(a_1)_0] = 0 \in \operatorname{gr} A^\delta$. Hence $[(f)_d] \cdot [(g_1)_{-d}] = [(1)_0] \in \operatorname{gr} A^\delta$. Consequently $[(f)_d]$ is a unit in $\operatorname{gr} A^\delta$, as required.

It remains to consider the case of $k = 0$. In this case $a_0 = fg_2$, with element $g_2 := 1 - a_2 f - a_3 f^2 - \cdots - a_l f^{l-1}$. Then $\delta(g_2) = \delta(a_0) - \delta(f) < w - d < 0$ and $g_2 = 1 - fg_1$ with $g_1 := a_2 + a_3 f + \cdots + a_l f^{l-2}$. Since $\delta(g_2) < 0 = \delta(1)$, it follows that $\delta(fg_1) = \delta(1) = 0$, and therefore $\delta(g_1) = -\delta(f) = -d$. Consequently $[(f)_d] \cdot [(g_1)_{-d}] = [(1)_0] - [(g_2)_0] = [(1)_0] \in \operatorname{gr} A^\delta$, implying $[(f)_d]$ is a unit in $\operatorname{gr} A^\delta$, which completes the proof of 1(c).

Next we prove assertion 2. Assume δ is a semidegree. Due to remark 3.2.3 it suffices to consider the case when $[(f)_{\delta(f)}]$ is not a unit in $\operatorname{gr} A^\delta$. Then $\delta(f) = w$ according to assertion 1(c). We start with introducing two surjective $\mathbb{K}$-algebra homomorphisms $\phi : A[s] \twoheadrightarrow A$ and $\Phi : A[s]^{\delta_e} \twoheadrightarrow A^{\tilde{\delta}}$ by means of formulae

$$\phi(\sum_{i=1}^{k} a_i s^i) := \sum_{i=1}^{k} a_i f^i \quad \text{for any } a_1, \ldots, a_k \in A,$$

$$\Phi((H)_d) := (\phi(H))_d \quad \text{for all } H \in A[s],\ d \geq \delta_e(H) \in \mathbb{Z}.$$

Clearly ϕ is surjective and $\ker \phi = J$. It follows that Φ is a surjective homomorphism of graded rings with $\ker \Phi = J^{\delta_e}$, and consequently, $A^{\tilde{\delta}} = A[s]^{\delta_e}/J^{\delta_e}$. Moreover, δ being a semidegree on A implies that δ_e is also a semidegree on $A[s]$ (lemma 3.2.1) and therefore $J^{\delta_e} = \langle (s-f)_{\delta_e(s-f)} \rangle$ (lemma 2.2.22), which completes the proof of 2(a).

Next we prove assertion 2(b). Ring $\operatorname{gr} A^{\tilde{\delta}} = A^{\tilde{\delta}}/\langle (1)_1 \rangle \cong A[s]^{\delta_e}/(J^{\delta_e} + \langle (1)_1 \rangle)$ because $A^{\tilde{\delta}} = A[s]^{\delta_e}/J^{\delta_e}$. The element z in the assertion of 2(b) (which corresponds to $[(f)_w]$) is precisely the equivalence class $[(s)_w]$ of $(s)_w \in A[s]^{\delta_e}$. Note that the homomorphism defined by $A^{\delta}[s] \ni \sum (f_i)_{d-iw} s^i \mapsto (\sum f_i s^i)_d \in A[s]^{\delta_e}$ is an isomorphism. Since $\delta_e(s) = w < \delta(f) = \delta_e(f)$, it follows that $(s-f)_{\delta_e(s-f)} = (s)_w (1)_{\delta(f)-w} + (f)_{\delta(f)}$ and therefore $J^{\delta_e} + \langle (1)_1 \rangle = \langle (s-f)_{\delta_e(s-f)}, (1)_1 \rangle = \langle (f)_{\delta(f)}, (1)_1 \rangle$. Hence $\operatorname{gr} A^{\tilde{\delta}} \cong A^{\delta}[s]/\langle (f)_{\delta(f)}, (1)_1 \rangle = (A^{\delta}/\langle (f)_{\delta(f)}, (1)_1 \rangle)[s]$. But $A^{\delta}/\langle (f)_{\delta(f)}, (1)_1 \rangle \cong (A^{\delta}/\langle (1)_1 \rangle)/(\langle (f)_{\delta(f)}, (1)_1 \rangle / \langle (1)_1 \rangle)$ and $\langle (f)_{\delta(f)}, (1)_1 \rangle / \langle (1)_1 \rangle$ is precisely the ideal generated by $[(f)_{\delta(f)}]$ in $\operatorname{gr} A^{\delta}$, which is $\operatorname{gr} \mathfrak{a}$, while $A^{\delta}/\langle (1)_1 \rangle \cong \operatorname{gr} A^{\delta}$. It follows that $\operatorname{gr} A^{\tilde{\delta}} \cong (\operatorname{gr} A^{\delta}/\operatorname{gr} \mathfrak{a})[s]$, and completes the proof of assertion 2(b).

It remains only to prove assertion 2(c). Due to assertion 2(b), $\operatorname{gr} \mathfrak{a}$ is a prime ideal of $\operatorname{gr} A^{\delta}$ iff $\operatorname{gr} A^{\tilde{\delta}}$ is a domain and, of course, iff $\langle (1)_1 \rangle$ is a prime ideal of $A^{\tilde{\delta}}$. But $\langle (1)_1 \rangle$ is a prime ideal of $A^{\tilde{\delta}}$ iff $\tilde{\delta}$ is a semidegree (theorem 2.2.1), which completes the proof of the theorem. □

Theorem 3.2.4 motivates the following:

Definition. Let δ be a semidegree on A. The *leading form* $\mathfrak{L}_{\delta}(f)$ of an element f of A is the equivalence class $[(f)_{\delta(f)}]$ of $(f)_{\delta(f)}$ in $\operatorname{gr} A^{\delta}$.

If δ is a semidegree on A and the ideal $\langle \mathfrak{L}_\delta(f)\rangle$ of $\operatorname{gr} A^\delta$ generated by the leading form $\mathfrak{L}_\delta(f)$ of $f \in A$ is prime, then $\tilde{\delta}$ is also a semidegree on A (theorem 3.2.4). Semidegree $\tilde{\delta}$ differs from δ according to assertion 1. of theorem 3.2.4. On the other hand, assertion 1(b) of theorem 3.2.4 shows that $\tilde{\delta}$ agrees with δ off $\operatorname{gr}\mathfrak{a}$. We will say that $\tilde{\delta}$ is formed by the *iteration procedure* starting with semidegree δ by means of $f \in A$.

Example 3.2.5. Let $A := \mathbb{K}[x_1, x_2]$ and δ be the semidegree on A defined in 2.1.2. Recall that $\delta(x_1) = 3$, $\delta(x_2) = 2$ and $\delta(x_1^2 - x_2^3) = 1$. Moreover, $\mathbb{K}$-algebra A^δ coincides with $\mathbb{K}[(1)_1, (x_1)_3, (x_2)_2, (x_1^2 - x_2^3)_1] = \mathbb{K}[X_1, X_2, Y, Z]/\langle YZ^5 - X_1^2 + X_2^3\rangle$. We claim that δ is formed by an iteration procedure by means of $f := x_1^2 - x_2^3$ starting with the weighted degree η which assigns weight 3 to x_1 and 2 to x_2. Indeed, $\operatorname{gr} A^\eta \cong \mathbb{K}[x_1, x_2]$ via the map that sends $\mathfrak{L}_\eta(h) \in \operatorname{gr} A^\eta$ to the leading weighted homogeneous component of h. Then, since $f = x_1^2 - x_2^3$ is weighted homogeneous, $\mathfrak{L}_\eta(f) = f$. Since $\mathfrak{L}_\eta(f) = f \in \mathbb{K}[x_1, x_2] = \operatorname{gr} A^\eta$, it follows that ideal $\langle \mathfrak{L}_\eta(f)\rangle$ is prime. Therefore according to assertion 2. of theorem 3.2.4, degree like function $\tilde{\eta}$ formed by the iteration procedure by means of f starting with η is in fact a semidegree. Also $A^{\tilde{\eta}} = A[s]^{\eta_e}/\langle (s-f)_6\rangle$, where η_e is the weighted degree on $A[s]$ that extends η and sends s to 1, as defined in the paragraph preceding lemma 3.2.1. Then with $t := (1)_1$, $A[s]^{\eta_e}/\langle (s - x_1^2 + x_2^3)_6 = \mathbb{K}[x_1, x_2, s, t]/\langle st^5 - x_1^2 + x_2^3\rangle \cong A^\delta$. To summarize, semidegree δ of example 2.1.2 coincides with the iterated semidegree $\tilde{\eta}$.

Remark 3.2.6. Assume a semidegree δ on the coordinate ring A of an affine variety X is constructed by means of finitely many iterations starting with a semidegree η. Denote by X^η and X^δ the completions of the d-uple embedding of X into appropriate projective spaces (valid for appropriate $d \in \mathbb{Z}_+$ [27, Lemma in section III.8]). Then we can express the degree of X^δ in terms of the degree of X^η (in a straightforward generalization of theorem 3.2.7 below). In particular, in the special case of η being a weighted homogeneous degree on $A := \mathbb{K}[x_1, \ldots, x_n]$ with weights $0 < d_i := \eta(x_i)$, $1 \leq i \leq n$, $\deg X^\eta = \frac{1}{d_1 \cdot \cdots \cdot d_n}$ (example 3.1.4) and an explicit formula for $D := \deg X^\delta$, which appears in the affine Bezout-type theorem 3.1.1, follows:

Theorem 3.2.7 (see [25, Example 9] and [26, Theorem 3.1.5])**.** *Let δ_0 be a weighted degree on $A := \mathbb{K}[x_1, \ldots, x_n]$. Let $k \geq 1$ and for each $i = 1, \ldots, k$, let δ_i be a semidegree on A obtained by an iteration procedure starting with δ_{i-1} by means of a polynomial h_i (with $\langle \mathfrak{L}_{\delta_{i-1}}(h_i) \rangle$ being prime in $\operatorname{gr} A^{\delta_{i-1}}$) by assigning to the polynomial h_i a weight w_i with $0 < w_i < \delta_{i-1}(h_i)$. Then*

$$\frac{D}{d^n} = \frac{\delta_0(h_1) \cdots \delta_{k-1}(h_k)}{\delta_0(x_1) \cdots \delta_0(x_n) w_1 \cdots w_k} , \tag{B}$$

where $D := \deg X^\delta$ and $d \in \mathbb{Z}_+$ are as in theorem 3.1.1 for $X := \mathbb{K}^n$, $A := \mathbb{K}[x_1, \ldots, x_n]$ and $\delta := \delta_k$.

Proof. Let $e_i := \delta_{i-1}(h_i)$ for $1 \leq i \leq k$. According to assertion 2. of theorem 3.2.4, rings $A^{\delta_i} \cong (A^{\delta_{i-1}}[s_i])^{\delta^e_{i-1}} / \langle (s_i - h_i)_{e_i} \rangle$, where s_i are indeterminates and δ^e_{i-1} extend δ_{i-1} by assigning weights w_i to s_i. It follows by induction on i with $x_0 = (1)_1$ that

$$A^{\delta_i} = \mathbb{K}[x_0, \ldots, x_n, s_1, \ldots, s_i]/J_i ,$$

where $J_i := \langle \tilde{h}_1 - x_0^{e_1 - w_1} s_1, \ldots, \tilde{h}_i - x_0^{e_i - w_i} s_i \rangle$, $1 \leq i \leq k$, and $\tilde{h}_j \in \mathbb{K}[\bar{x}, \bar{s}]$ are weighted homogeneous polynomials in $(\bar{x}, \bar{s}) := (x_0, \ldots, x_n, s_1, \ldots, s_{j-1})$ whose equivalence classes in A^{δ_i} are $(h_j)_{e_j}$, $1 \leq j \leq i$.

Let $\tilde{\delta}$ be the weighted degree on $R_k := \mathbb{K}[x_0, \ldots, x_n, s_1, \ldots, s_k]$ which assigns weight 1 to x_0, $d_i := \delta_0(x_i)$ to x_i, $1 \leq i \leq n$, and w_j to indeterminates s_j, $1 \leq j \leq k$. Then homomorphism $\pi : R_k/J_k \to A^\delta$ of graded $\mathbb{K}$-algebras is surjective, and, therefore, for each $f \in A$, there is a polynomial $\tilde{f}$ in R_k with $\tilde{\delta}(\tilde{f}) = \delta(f)$ and $\tilde{f} \mapsto (f)_{\delta(f)}$ under homomorphism π. Moreover, homomorphism π induces an embedding of X^{δ_k} into the weighted projective space $\mathbf{W}\mathbb{P} := \mathbb{P}^{n+k}(\mathbb{K}; 1, d_1, \ldots, d_n, w_1, \ldots, w_k)$. Since J_k is generated by exactly k polynomials in R_k, it follows that the image of X^δ in $\mathbf{W}\mathbb{P}$ is a *complete intersection.* Identifying X^δ with its image in $\mathbf{W}\mathbb{P}$, it follows that $X^\delta = V(J_k)$ and, therefore, that for any $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$

$$\bigcap_{i=1}^{n} \{x \in \mathbb{K}^n : f_i(x) = 0\} \subseteq X^\delta \cap V(\tilde{f}_1) \cap \cdots \cap V(\tilde{f}_n) = \quad (3.5)$$

$$V(\tilde{h}_1 - x_0^{e_1 - w_1} s_1) \cap \cdots \cap V(\tilde{h}_k - x_0^{e_k - w_k} s_k) \cap V(\tilde{f}_1) \cap \cdots \cap V(\tilde{f}_n).$$

Arguing via an embedding of $\mathbf{W}\mathbb{P} \hookrightarrow \mathbb{P}^N(\mathbb{K})$ it suffices to choose as f_j's the pull backs of generic linear polynomials on $\mathbb{P}^N(\mathbb{K})$ and then the intersection on the right hand side of the equality in (3.5) would consist of isolated points in $X := \mathbb{K}^n \hookrightarrow X^\delta \hookrightarrow \mathbb{P}^N(\mathbb{K})$ (of multiplicities one and of the total number being the degree of X^δ in $\mathbb{P}^N(\mathbb{K})$ according to the commonly used geometric definition of degree of a projective variety [13, Definition 18.1]). Consequently due to weighted homogeneous Bezout theorem (example 3.1.4) on $\mathbb{K}^{n+k}$, the sum of the intersection multiplicitiess of Cartier divisors corresponding to $\tilde{h}_i - x_0^{e_i - w_i} s_i$, $1 \leq i \leq k$, and $\tilde{f}_j$, $1 \leq j \leq n$, at the points in the left hand side of the equality in (3.5) is $\frac{\delta(f_1)\cdots\delta(f_n) e_1 \cdots e_k}{d_1 \cdots d_n w_1 \cdots w_k}$, and by the Bezout theorem for semidegrees, the sum of the multiplicities of the fiber $f^{-1}(0)$ at the points in the left hand side of the inclusion in (3.5) is $\frac{D}{d^n}\delta(f_1)\cdots\delta(f_n)$. Formula (B) follows by comparing these two expressions, which completes the proof. □

Example 3.2.8. Let $f_k := (x_1 + (x_1^2 - x_2^3)^2, (x_1^2 - x_2^3)^k) : \mathbb{K}^2 \to \mathbb{K}^2$. We estimate the size of fibers of $f_k =: (f_{k1}, f_{k2})$ in three different ways. The first one is by means of the weighted homogeneous Bezout formula (3.3). It is straightforward to see that the smallest upper bound given by (3.3) for f_k is achieved for $d_1 = 3p$ and $d_2 = 2p$ for some $p \geq 1$, in which case the bound is $\frac{12p \cdot 6kp}{3p \cdot 2p} = 12k$. The second approach we take is via Bernstein's theorem (see section 0.2). Let $a := (a_1, a_2) \in \mathbb{K}^2$ and let $\mathcal{P}$ and $\mathcal{Q}_k$ be the Newton polygons of $x_1 + (x_1^2 - x_2^3)^2 - a_1$ and, respectively, of $(x_1^2 - x_2^3)^k - a_2$. The BKK bound for $|f_k^{-1}(a)|$ for non-zero a_1, a_2 is then $2\mathcal{M}(\mathcal{P}, \mathcal{Q}) = \mathrm{Vol}(\mathcal{P} + \mathcal{Q}) - \mathrm{Vol}(\mathcal{P}) - \mathrm{Vol}(\mathcal{Q}) = \frac{1}{2}(2k+4)(3k+6) - 12 - 3k^2 = 12k$. (A similar argument implies that the BKK bound for a_1 or a_2 being zero is as well $12k$.)

Let δ be the iterated semidegree on $\mathbb{K}[x_1, x_2]$ from example 3.2.5, so that $\delta(x_1) = 3$,

$\delta(x_2) = 2$ and $\delta(x_1^2 - x_2^3) = 1$. Then $D/d^n = \frac{6}{3\cdot 2\cdot 1} = 1$ (theorem 3.2.7). The estimate of $|f_k^{-1}(a)|$ given by (A) with this δ is then $\delta(x_1 + (x_1^2 - x_2^3)^2)\delta((x_1^2 - x_2^3)^k) = 3k$. Moreover, ideals generated by $(f_{k1})_{\delta(f_{k1})}$, $(f_{k2})_{\delta(f_{k2})}$ and $(1)_1$ in A^δ are primary to the irrelevant ideal of A^δ and, therefore, completion ψ_δ preserves f_k at ∞ over all points in $\mathbb{K}^2$. In other words our bound with iterated semidegree δ is exact!

3.3 A Formula for Subdegrees

Assume $f := (f_1, \ldots, f_n) : X \to \mathbb{K}^n$ is a dominating morphism of affine varieties with generically finite fibers and $\delta := \max\{\delta_j : 1 \leq j \leq N\}$ is a complete subdegree on $A := \mathbb{K}[X]$, i.e. δ is non-negative, finitely generated and $\delta^{-1}(0) = \mathbb{K} \setminus \{0\}$. Assume $\delta_j(f_i) > 0$ for each i, j. In the spirit of theorem 3.1.1 we derive in this section an upper bound for the number of points in a generic fiber of f in X (counted with multiplicity) in terms of the degree of a projective completion of X.

Definition.

- Let $g \in A$ and $\operatorname{div}_X(g)$ be the principal Cartier divisor corresponding to g on X. Assume that the corresponding Weil divisor is $[\operatorname{div}_X(g)] = \sum r_i[V_i]$. Given a completion $X \hookrightarrow Y$ of X, we write $[\overline{\operatorname{div}}_X^Y(g)]$ for the Weil divisor on Y given by:

$$[\overline{\operatorname{div}}_X^Y(g)] := \sum r_i[\overline{V}_i],$$

where $\overline{V}_i$ is the closure of V_i on Y. If $Y = X^\delta$ for some degree like function δ on A, then we also make use of notation $\overline{\operatorname{div}}_X^\delta(g)$ for $\overline{\operatorname{div}}_X^Y(g)$.

- If $g \in A$ is such that $\delta_j(g) > 0$ for all $j = 1, \ldots, N$, then

$$\delta_g := e_g \cdot \max\{\frac{\delta_j}{\delta_j(g)} : 1 \leq j \leq N\},$$

where e_g is a suitable integer to ensure that δ_g is integer valued (e.g. one can take $e_g := \prod_{j=1}^N \delta_j(g)$).

Let the filtration corresponding to δ be $\mathcal{F} := \{F_d : d \geq 0\}$. Identify A^δ with $\sum_{d\in\mathbb{Z}} F_d t^d$. Recall that $X^\delta := \operatorname{Proj} A^\delta$ is the union of affine charts of the form $\operatorname{Spec} A^\delta_{(gt^d)}$, where $d > 0$ and $A^\delta_{(gt^d)}$ is the subring of elements of degree zero of the localizations $A^\delta_{gt^d}$. Say $U_0, \ldots, U_m$ is an open cover of X^δ with $U_j := \operatorname{Spec} A^\delta_{(g_j t^{l_j})}$ for some $l_j \geq 1$ and $g \in F_{l_j}$ for every j. Moreover, assume that $g_0 = 1$ and $l_0 = 1$, so that $U_0 = \operatorname{Spec} A^\delta_{(t)} = \operatorname{Spec} A$. Let d be a common multiple of $l_1, \ldots, l_m$. Then for each j, $h_j := \frac{t^d}{(g_j)^{d/l_j} t^d}$ is a regular function on U_j and h_j/h_k is a unit on $U_j \cap U_k$. Therefore collection $\{(h_j, U_j)\}_j$ defines an *effective* Cartier divisor $D^\delta_{d,\infty}$ on X^δ, which we call the *d-uple divisor at infinity*. Its associated Weil divisor is

$$[D^\delta_{d,\infty}] := \sum_{j=1}^{N} \operatorname{ord}_j(D^\delta_{d,\infty})[V_j],$$

where $V_1, \ldots, V_N$ are the irreducible components of X_∞ and ord_j is the shorthand for ord_{V_j} (where ord_{V_j} is as defined in section 2.0.2). Support of $[D^\delta_{d,\infty}]$ being X_∞ justifies index ∞ as a subscript of $D^\delta_{d,\infty}$.

Lemma 3.3.1. *Let X, A, δ and $D^\delta_{d,\infty}$ be as above. Then*

1. $[D^\delta_{d,\infty}] = \sum_{j=1}^N \frac{d}{d_j}[V_j]$ *where for every j, integer d_j is the positive generator of the subgroup of $\mathbb{Z}$ generated by $\{\delta_j(f) : f \in A\}$.*
2. *Let $g \in A$ be such that δ_g is finitely generated. Then the principal divisor of g^d on X^{δ_g} is $[\operatorname{div}_{X^{\delta_g}}(g^d)] = d[\overline{\operatorname{div}_X}^{\delta_g}(g)] - e_g[D^{\delta_g}_{d,\infty}]$.*

Proof. 1. Fix integer j, $1 \leq j \leq N$. Local ring $\mathcal{O}_{V_j, X^\delta}$ is a discrete valuation ring and its associated valuation is $\nu_j(\cdot) := -\frac{\delta_j(\cdot)}{d_j}$ (proposition 2.2.12). Pick k, $1 \leq k \leq N$, such that $V_j \cap U_k \neq \emptyset$. Recall that $U_k := \operatorname{Spec} A^\delta_{(g_k t^{l_k})}$ and a local equation for $D^\delta_{d,\infty}$ on U_k is $\frac{t^d}{(g_k)^{d/l_k} t^d}$. Let $\mathfrak{p}_j$ be the ideal of A^δ corresponding to V_j. Since $V_j \cap U_k \neq \emptyset$, it follows that $g_k t^{l_k} \notin \mathfrak{p}_j$ and therefore $\delta_j(g_k) = l_k$ according to assertion 1 of lemma 2.2.1.2. Therefore $\operatorname{ord}_j(D^\delta_{d,\infty}) = \nu_j(\frac{t^d}{(g_k)^{d/l_k} t^d}) = \nu_j(1/g_k^{d/l_k}) = -\frac{d}{l_k}\nu_j(g_k) = \frac{d}{l_k} \cdot \frac{l_k}{d_j} = \frac{d}{d_j}$. It follows that $[D^\delta_{d,\infty}] := \sum_{j=1}^N \operatorname{ord}_j(D^\delta_{d,\infty})[V_j] = \sum_{j=1}^N \frac{d}{d_j}[V_j]$, which completes the proof of assertion 1.

2. Reindexing the δ_j's if necessary, we may assume that the minimal presentation of δ_g is $\delta_g = \max\{\frac{e_g \cdot \delta_j}{\delta_j(g)} : 1 \leq j \leq M\}$ for some $M \leq N$. For $1 \leq j \leq M$, let V'_j be the irreducible component of $X^{\delta_g}_\infty$ corresponding to δ_j and d'_j be the positive generator of the subgroup of $\mathbb{Z}$ generated by $\{\frac{e_g \cdot \delta_j(h)}{\delta_j(g)} : h \in A\}$. Then, as a straightforward consequence of proposition 2.2.12 it follows that $\operatorname{ord}_{V'_j}(g^d) = -\frac{e_g d}{d'_j}$ for every j and therefore $[\operatorname{div}_{X^{\delta_g}}(g^d)] = d[\overline{\operatorname{div}_X}^{\delta_g}(g)] + \sum_{j=1}^M \operatorname{ord}_{V'_j}(g^d)[V'_j] = d[\overline{\operatorname{div}_X}^{\delta_g}(g)] - \sum_{j=1}^M \frac{e_g d}{d'_j}[V'_j]$. On the other hand, applying assertion 1 with δ_g in place of δ yields that $[D^{\delta_g}_{d,\infty}] = \sum_{j=1}^M \frac{d}{d'_j}[V'_j]$. Therefore $[\operatorname{div}_{X^{\delta_g}}(g^d)] = d[\overline{\operatorname{div}_X}^{\delta_g}(g)] - e_g[D^{\delta_g}_{d,\infty}]$, as required. □

Theorem 3.3.2. *Assume that* $f := (f_1, \ldots, f_n) : X \to \mathbb{K}^n$ *and* $\delta := \max\{\delta_j : 1 \leq j \leq N\}$ *are as in the first paragraph of this section and that* $\delta_j(f_i) > 0$, $1 \leq i, j \leq n$. *Assume also that for* $i = 1, \ldots, n$, *subdegrees* δ_{f_i} *are finitely generated. Let* $d_{f_i} \geq 1$, $1 \leq i \leq n$, *be such that the* d_{f_i}*-uple embedding of* $X^{\delta_{f_i}}$ *is a closed immersion of* $X^{\delta_{f_i}}$ *into a projective space* $\mathbb{P}^{L_i}(\mathbb{K})$. *Let* $L := \prod_{i=1}^n (L_i + 1) - 1$ *and* $\bar{X}$ *be the closure of the image of* X *in* $\mathbb{P}^L(\mathbb{K})$ *under the composition of the following maps:*

$$X \hookrightarrow X^{\delta_{f_1}} \times \cdots \times X^{\delta_{f_n}} \hookrightarrow \mathbb{P}^{L_1}(\mathbb{K}) \times \cdots \times \mathbb{P}^{L_n}(\mathbb{K}) \hookrightarrow \mathbb{P}^L(\mathbb{K}),$$

where the first map is the diagonal embedding and the last map is the Segre embedding. Then for all $a \in \mathbb{K}^n$,

$$|f^{-1}(a)| \leq \frac{e_{f_1} \cdots e_{f_n}}{n^n d_{f_1} \cdots d_{f_n}} \deg(\bar{X}), \tag{C}$$

where $|f^{-1}(a)|$ *is the number of the isolated points in* $f^{-1}(a)$ *each counted with the multiplicity of* $f^{-1}(a)$ *at the respective point.*

Question: Is it true that the completeness of δ implies the finite generation property of every δ_{f_i}?

Proof of theorem 3.3.2. Denote by $[y_{i,0} : \cdots : y_{i,L_i}]$ the homogeneous coordinates on $\mathbb{P}^{L_i}(\mathbb{K})$. Without loss of generality we may assume that $X^{\delta_{f_i}}_\infty = X^{\delta_{f_i}} \cap V(y_{i,0})$.

Denote by X^η the closure of the diagonal embedding of variety X into the product $X^{\delta_{f_1}} \times \cdots \times X^{\delta_{f_n}} \subseteq \mathbb{P}^{L_1}(\mathbb{K}) \times \cdots \times \mathbb{P}^{L_n}(\mathbb{K}) =: Y$. Denote by $[y_0 : \cdots : y_L : z_1 : \cdots : z_n]$ the homogeneous coordinates on $\mathbb{P}^{L'}(\mathbb{K})$, where $L' := L + n$. Let us identify $\mathbb{P}^L(\mathbb{K})$ with the subspace $V(z_1, \cdots, z_n)$ of $\mathbb{P}^{L'}(\mathbb{K})$ and let $s : \mathbb{P}^{L_1}(\mathbb{K}) \times \cdots \times \mathbb{P}^{L_n}(\mathbb{K}) \to \mathbb{P}^L(\mathbb{K})$ denote the Segre embedding. Let $s' : X^\eta \to \mathbb{P}^{L'}(\mathbb{K})$ be the map defined by:

$$s' :\ X^\eta \ni ([y_{1,0} : \cdots : y_{1,L_1}], \ldots, [y_{n,0} : \cdots : y_{n,L_n}]) \mapsto [s(y) : (y_{1,0})^n : \cdots : (y_{n,0})^n] \in \mathbb{P}^{L'}(\mathbb{K}).$$

Fix an i, $1 \le i \le n$. Let $D_i := \pi_i^*(D^{\delta_{f_i}}_{d_{f_i},\infty})$, where $\pi_i : X^\eta \to X^{\delta_{f_i}}$ is the projection onto the i-th factor of Y. Due to our choice of $y_{i,0}$, Cartier divisor $D^{\delta_{f_i}}_{d_{f_i},\infty}$ is precisely the restriction of the divisor of $y_{i,0}$ to $X^{\delta_{f_i}}$. It follows that $s'^*(D'_i) = nD_i$, where D'_i is the restriction of the divisor of z_i to $s'(X^\eta)$.

Then $(D_1, \ldots, D_n) = \frac{1}{n^n}(s'^*(D'_1), \ldots, s'^*(D'_n)) = \frac{1}{n^n}(D'_1, \ldots, D'_n)$, since intersection numbers are preserved under the pull backs by proper birational morphisms [10, Example 2.4.3]. Since each D'_i is the divisor of a linear form on $s'(X^\eta)$, it follows that the intersection number $(D'_1, \ldots, D'_n) = \deg s'(X^\eta)$. On the other hand, $s|_{X^\eta} = \pi' \circ s'$, where π' is the projection onto the first L coordinates. Since the mapping degree of $\pi'|_{s'(X^\eta)}$ is 1, it follows that $\deg(s(X^\eta)) = \deg s'(X^\eta)$ [28, Proposition 5.5]. Combining three equalities established in this paragraph it follows that $(D_1, \ldots, D_n) = \frac{1}{n^n} \deg s(X^\eta)$.

Let $E_i := \pi_i^*(\overline{\mathrm{div}}^{\delta_{f_i}}_X(f_i))$. According to assertion 2 of lemma 3.3.1 $[\mathrm{div}_{X^{\delta_{f_i}}}(f_i^{d_{f_i}})] = d_{f_i}[\overline{\mathrm{div}}^{\delta_{f_i}}_X(f_i)] - e_{f_i}[D^{\delta_{f_i}}_{d_{f_i},\infty}]$. Since $\pi_i^*(\mathrm{div}_{X^{\delta_{f_i}}}(f_i)) = \mathrm{div}_{X^\eta}(f_i)$ it follows that

$$(E_1, \ldots, E_n) = \frac{e_{f_1} \cdots e_{f_n}}{d_{f_1} \cdots d_{f_n}}(D_1, \ldots, D_n) = \frac{e_{f_1} \cdots e_{f_n}}{n^n d_{f_1} \cdots d_{f_n}} \deg s(X^\eta)\ .$$

Fix an i, $1 \le i \le n$. Note that $D^{\delta_{f_i}}_{d_{f_i},\infty}$ are *very ample*, i.e. are the pull backs of the hyperplane sections under the embeddings of $X^{\delta_{f_i}}$ into $\mathbb{P}^{L_i}(\mathbb{K})$. In particular, these divisors are *base point free* [14, Section II.7]. Then the pull backs D_i of $D^{\delta_{f_i}}_{d_{f_i},\infty}$ are also base point free, and so are $e_{f_i}E_i$ (the latter being linearly equivalent to $d_{f_i}D_i$). Also, since E_i's are *effective* (defined in section 2.0.2), it follows that the intersection number $(E_1, \ldots, E_n)$ bounds the sum of the intersection multiplicities of E_i's at the isolated points of the intersection $\bigcap_{i=1}^n \operatorname{Supp}(E_i)$ [10, Section 12.2]. Of course $X \cap (\bigcap_{i=1}^n \operatorname{Supp}(E_i)) = f^{-1}(0)$. Therefore $|f^{-1}(0)| \leq (E_1, \ldots, E_n)$, which completes the proof of the theorem. □

Future plans:

1. Let $f := (f_1, \ldots, f_n) : X \to \mathbb{K}^n$ be any generically finite map (not necessarily satisfying the hypotheses of theorem 3.3.2). Replacing f by $\xi \circ f$ for a generic affine transformation ξ of $\mathbb{K}^n$ (and reordering δ_j's if necessary), one may assume that there is an $M \leq N$ such that

(a) $\delta_j(f_i) > 0$ for all i and all $j = 1, \ldots, M$, and

(b) $\delta_j(f_i) = 0$ for all i and all $j = M+1, \ldots, N$.

We expect that it should be possible to extend theorem 3.3.2 in this setting as a consequence of extending the arguments of the proof of theorem 3.3.2 to the case of $\tilde{X} := \operatorname{Spec} \tilde{A}$, where $\tilde{A} := \{g \in A : \delta_j(g) \leq 0 \text{ for } M+1 \leq j \leq N\}$, and for the completion $\tilde{X}^{\tilde{\delta}}$ of $\tilde{X}$ determined by $\tilde{\delta} := \max\{\delta_j : 1 \leq j \leq M\}$.

2. Moreover, we hope to prove (by means of an extension of our theorem A.1 to the case of subdegrees) that for completion X^δ that preserves map f at ∞ (in the setting of theorem 3.3.2), inequality in (C) can be replaced by equality.

Appendix A

Non-degeneracy condition in Bernstein's theorem

Starting with a precise statement of Bernstein's theorem, we provide a proof of our interpretation of Kushnirenko-Bernstein's non-degeneracy condition (E3) as

$$\text{toric completion } X_{\mathcal{P}} \text{ of } (\mathbb{C}^*)^n \text{ preserves } \{f_1, \ldots, f_n\} \text{ at } \infty \text{ over } 0, \tag{E}$$

where $X_{\mathcal{P}}$ is the toric variety associated with the *Minkowski sum* $\mathcal{P}$ of the Newton polytopes of components of f.

Theorem (Bernstein [2])**.** *For each $i = 1, \ldots, n$, let A_i be a finite subset of $\mathbb{Z}^n$ and f_i be a Laurent polynomial in $\mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ such that* $\operatorname{Supp}(f_i) \subseteq A_i$*. Then*

$$|V(f_1, \ldots, f_n)| \leq n!\mathcal{M}(A_1, \ldots, A_n) \tag{$*$}$$

where $|V(f_1, \ldots, f_n)|$ is the number of isolated points in the set $V(f_1, \ldots, f_n) := \{x \in (\mathbb{C}^*)^n : f_1(x) = \cdots = f_n(x) = 0\}$ each counted with the multiplicity of $f^{-1}(0)$ at the respective point, and $\mathcal{M}(A_1, \ldots, A_n)$ is the mixed volume *of $A_1, \ldots, A_n$. Let $f_i = \sum_{\beta \in A_i} a_{i,\beta} x^\beta$, and for each $\alpha \in (\mathbb{Z}^n)^*$, let $A_{i,\alpha} := \{\beta \in A_i : \langle \alpha, \beta \rangle \leq \langle \alpha, \gamma \rangle \text{ for all } \gamma \in A_i\}$ and $f_{i,\alpha} := \sum_{\beta \in A_{i,\alpha}} a_{i,\beta} x^\beta$. Inequality ($*$) holds with an equality iff*

$$\textit{for all } \alpha \in \mathbb{Z}^n,\ V(f_{1,\alpha}, \ldots, f_{n,\alpha}) = \emptyset. \tag{$**$}$$

Let f_i's and A_i's be as in Bernstein's theorem. Let $\mathcal{P}$ be the convex hull of $A_1+\cdots+A_n$. Recall from section 2.0.1 that if $\dim \mathcal{P} = n$, then there is an embedding $\phi_{\mathcal{P}} : (\mathbb{C}^*)^n \hookrightarrow X_{\mathcal{P}}$, where $X_{\mathcal{P}}$ is the toric variety corresponding to $\mathcal{P}$.

Theorem A.1 (see [26, Appendix])**.** *Assume* $\dim \mathcal{P} = n$. *Then* $(**)$ *holds iff property* (E) *takes place.*

Proof. Recall that if $\psi : \mathbb{Z}^n \to \mathbb{Z}^n$ is an injective affine transformation, then $X_{\mathcal{P}} \cong X_{\psi(\mathcal{P})}$ (theorem 2.0.2). Moreover, for $g := \sum a_\beta x^\beta$, the image of $V(g) \cap (\mathbb{C}^*)^n$ under the above isomorphism is $V(g_\psi) \cap (\mathbb{C}^*)^n$, where $g_\psi := \sum a_\beta x^{\psi(\beta)}$. For $f := (f_1, \ldots, f_n) : (\mathbb{C}^*)^n \to \mathbb{C}^n$, we also define $f_\psi := (f_{1\psi}, \ldots, f_{n\psi})$. It follows that for either one of the properties (asserted to be equivalent in the statement of the theorem), f satisfies the property if and only if so does f_ψ. Therefore we may without loss of generality replace in the course of this proof f by f_ψ for any injective affine transformation ψ of $\mathbb{Z}^n$.

We first prove the 'only if' implication. So, assume $\phi_{\mathcal{P}}$ does not preserve $\{f_1, \ldots, f_n\}$ at ∞ over 0. Let $\tilde{z} \in \overline{V(f_1)} \cap \cdots \cap \overline{V(f_n)} \cap X_\infty$. Recall that $X_\infty = \cup X_Q$, where the union is over *faces* Q of $P := \mathcal{P} \cap \mathbb{Z}^n$ (i.e. $Q = \mathcal{Q} \cap \mathbb{Z}^n$, where $\mathcal{Q}$ is a face of $\mathcal{P}$). Let $m\mathcal{P} \cap \mathbb{Z}^n := \{\alpha^0, \ldots, \alpha^N\}$, for an $m \geq n-1$ as in assertion 4 of theorem 2.0.2. Fix m. Then $X_{\mathcal{P}}$ is isomorphic to the closure of the image of $\phi_{\mathcal{P}} : (\mathbb{C}^*)^n \to \mathbb{P}^N$ which maps $x \mapsto [x^{\alpha^0} : \cdots : x^{\alpha^N}]$. Under this isomorphism, for a face $Q := \mathcal{Q} \cap \mathbb{Z}^n$ of P, X_Q is the closure of the image of the map $x \mapsto [\phi_{\mathcal{Q},0}(x) : \cdots : \phi_{\mathcal{Q},N}(x)]$, where

$$\phi_{\mathcal{Q},j}(x) = \begin{cases} x^{\alpha^j} & \text{if } \alpha^j \in m\mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Let Q be the *smallest* dimensional face of P such that $\tilde{z} \in X_Q$. Replacing f by f_ψ for a suitable injective affine transformation ψ of $\mathbb{Z}^n$ we may assume w.l.o.g. that

1. $A_i \subseteq (\mathbb{Z}_+)^n$ for each i, and
2. $0 \in Q \subseteq E_d := \mathbb{Z}\langle e^1, \ldots, e^d \rangle$, where $d := \dim(\operatorname{conv} Q)$ and $e^i \in \mathbb{Z}^n$, $1 \leq i \leq d$, have the j-th coordinate being 1 if $j = i$ and 0 otherwise.

(In particular, $P = \operatorname{conv}(A_1 + \cdots + A_n) \cap \mathbb{Z}^n \subseteq (\mathbb{Z}_+)^n$.) Reindexing the α^j's if necessary, we may also assume that $\alpha^0 = 0$, $\alpha^i \in Q$ for $1 \leq i \leq d$ and $\alpha^1, \ldots, \alpha^n$ are linearly independent. Let matrix U of size $n \times n$ be defined by

$$U := \begin{pmatrix} \alpha_1^1 & \cdots & \alpha_n^1 \\ \vdots & & \vdots \\ \alpha_1^n & \cdots & \alpha_n^n \end{pmatrix}.$$

The assumptions on α^j's imply that matrix U is 'block triangular', namely:

$$U = \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right),$$

where A is an invertible $d \times d$ matrix and C is an invertible $(n-d) \times (n-d)$ matrix over $\mathbb{Q}$. Let $V := U^{-1}$ and let $v^i := (v_1^i, \ldots, v_n^i)$ be the i-th row of V. It follows that

$$V = \left(\begin{array}{c|c} A^{-1} & 0 \\ \hline B' & C^{-1} \end{array}\right), \tag{A.1}$$

for an appropriate $(n-d) \times d$ matrix B'. Denote by $[z_0 : \cdots : z_N]$ the homogeneous coordinates on $\mathbb{P}^N$. Due to our choice of Q, $\tilde{z}_j \neq 0$ iff $\alpha^j \in \operatorname{conv}(mQ)$, in particular $\tilde{z}_0 \neq 0$.

Fix an $i \in \{1, \ldots, n\}$. Since $\tilde{z} \in \overline{V(f_i)}$, there is a curve $C_i \subseteq V(f_i)$ such that $\tilde{z} \in \overline{C_i}$. Let the j-th coordinate $(C_i(t))_j$ of the Puiseux expansion of $\overline{C}_i$ at $\tilde{z}$ be as follows (with index i suppressed in abuse, but for simplicity of notation):

$$(z_j/z_0)(t) = t^{\gamma_j}(b_{j,0} + b_{j,1}t^{\gamma_{j,1}} + b_{j,2}t^{\gamma_{j,2}} + \cdots) =: (C_i(t))_j,$$

where each γ_j is a non-negative rational number and $\{\gamma_{j,k}\}_k$ is an increasing sequence of positive rational numbers with a common denominator in $\mathbb{Z}_+$. Since $(z_j/z_0)(0) = \tilde{z}_j/\tilde{z}_0$ and $\tilde{z}_j \neq 0$ for $\alpha^j \in \operatorname{conv}(mQ)$, it follows that in fact

$$\gamma_j = 0 \text{ and } b_{j,0} = \tilde{z}_j/\tilde{z}_0 \tag{A.2}$$

whenever $\alpha^j \in \text{conv}(mQ)$, in particular for $1 \leq j \leq d$. Note that for all sufficiently small $t \in \mathbb{C}^*$, points $C_i(t) \in (\mathbb{C}^*)^n$, i.e. $(z_j/z_0)(t) = x^{\alpha^j}(t)/x^{\alpha^0}(t) = x^{\alpha^j}(t)$. Therefore, for $1 \leq j \leq n$,

$$\begin{aligned} x_j(t) := (C_i(t))_j &= x(t)^{e^j} \\ &= x(t)^{v_1^j\alpha^1+\cdots+v_n^j\alpha^n} \\ &= \prod_{k=1}^n (x^{\alpha^k}(t))^{v_k^j} \\ &= t^{\tilde{\gamma}_j}(\tilde{b}_{j,0} + \tilde{b}_{j,1}t^{\tilde{\gamma}_{j,1}} + \tilde{b}_{j,2}t^{\tilde{\gamma}_{j,2}} + \cdots), \end{aligned} \tag{A.3}$$

where $\tilde{\gamma}_j := \sum_{k=1}^n v_k^j \gamma_k = \sum_{k=d+1}^n v_k^j \gamma_k$ are (not necessarily positive) rational numbers, $\{\tilde{\gamma}_{j,k}\}_k$ are increasing sequences of positive rational numbers with common denominators in $\mathbb{Z}_+$, and $\tilde{b}_{j,0} := \prod_{k=1}^n b_{k,0}^{v_k^j}$. Here for all j, k such that v_k^j is *not* an integer, we may define $b_{k,0}^{v_k^j}$ as follows: at first write $v_k^j = p_k^j/q_k^j$ with p_k^j and q_k^j being relatively prime and $q_k^j > 0$. Then fix for all curves C_i the choice of a q_k^j-th root of the respective $b_{k,0}$, and set $b_{k,0}^{v_k^j}$ to be the p_k^j-th power of that root.

Let $\tilde{\gamma} := (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n) \in \mathbb{Q}^n$ and $\tilde{b} := (\tilde{b}_{1,0}, \ldots, \tilde{b}_{n,0}) \in (\mathbb{C}^*)^n$. (Recall that $\tilde{\gamma} = \tilde{\gamma}(i)$ and $\tilde{b} = \tilde{b}(i) =: (\tilde{b}_{1,0}(i), \ldots, \tilde{b}_{n,0}(i))$ depend on the choice of curve C_i.) Since $\alpha^j \in (\mathbb{Z}_+)^n$ for every j, it follows (with once again index i in the right hand side of the following formula being suppressed) that

$$\phi_{\mathcal{P}}(C_i(t)) = [1 : (\tilde{b}^{\alpha^1} t^{\langle \tilde{\gamma}, \alpha^1 \rangle} + \text{ h.o.t.}) : \; \cdots \; : (\tilde{b}^{\alpha^N} t^{\langle \tilde{\gamma}, \alpha^N \rangle} + \text{ h.o.t.})],$$

where *h.o.t.* stands for the 'higher order terms' (in t). Since $\lim_{t\to 0} \phi_{\mathcal{P}}(C_i(t)) = \tilde{z}$, and since $\tilde{z}_j \neq 0$ iff $\alpha^j \in \text{conv}(mQ)$, it follows that $\langle \tilde{\gamma}, \alpha^j \rangle > 0$ if $\alpha^j \notin \text{conv}(mQ)$, and $\langle \tilde{\gamma}, \alpha^j \rangle = 0$ if $\alpha^j \in \text{conv}(mQ)$. But then Q is the 'face' of P corresponding to $\tilde{\gamma}$. Therefore $Q = \text{conv}(A_{1,\tilde{\gamma}} + \cdots + A_{n,\tilde{\gamma}}) \cap \mathbb{Z}^n$, where each $A_{j,\tilde{\gamma}}$ is the 'face' of A_j corresponding to $\tilde{\gamma}$ as defined in the statement of Bernstein's theorem. With all exponents $\tilde{\gamma}_j$, $\tilde{\gamma}_{j,k}$ and coefficients $\tilde{b}_{j,k}$, $1 \leq j \leq n$ and $k \geq 0$, from the expression for $C_i(t)$ in (A.3) depending on i, it follows

$$\begin{aligned} f_i(C_i(t)) &= f_i(t^{\tilde{\gamma}_1}\sum_{k\geq 0}\tilde{b}_{1,k}t^{\tilde{\gamma}_{1,k}},\ldots,t^{\tilde{\gamma}_n}\sum_{k\geq 0}\tilde{b}_{n,k}t^{\tilde{\gamma}_{n,k}}) \\ &= \sum_{\beta} a_{i,\beta}\prod_{j=1}^{n}(t^{\tilde{\gamma}_j}\sum_{k\geq 0}\tilde{b}_{j,k}t^{\tilde{\gamma}_{j,k}})^{\beta_j} \\ &= \sum_{\beta} a_{i,\beta}(\tilde{b}^{\beta}t^{\langle\tilde{\gamma},\beta\rangle} + \text{ h.o.t.}) \\ &= t^{d_i}(\sum_{\beta\in A_{i,\tilde{\gamma}}} a_{i,\beta}\tilde{b}^{\beta}) + \text{ h.o.t.}, \end{aligned}$$

for some $d_i \in \mathbb{Q}$. Since $f_i(C_i(t)) \equiv 0$, it follows that $f_{i,\tilde{\gamma}(i)}(\tilde{b}(i)) = \sum_{\beta\in A_{i,\tilde{\gamma}(i)}} a_{i,\beta}\tilde{b}(i)^{\beta} = 0$. Moreover, since $Q \subseteq E_d$, there exists $\tilde{\beta}(i) \in \mathbb{Z}^n$ such that $A_{i,\tilde{\gamma}(i)} \subseteq \tilde{\beta}(i) + E_d$. It follows that $\tilde{f}_i := x^{-\tilde{\beta}(i)}f_{i,\tilde{\gamma}(i)}$ depends only on coordinates $(x_1,\ldots,x_d)$. Hence for each $x = (x_1,\ldots,x_n) \in (\mathbb{C}^*)^n$, $f_{i,\tilde{\gamma}(i)}(x) = x^{\tilde{\beta}(i)}\tilde{f}_i(x_1,\ldots,x_d)$. Note that according to (A.1), $v_k^j = 0$ for $1 \leq j \leq d$ and $k > d$. Therefore (A.2) implies

$$\tilde{b}_{j,0}(i) = \prod_{k=1}^{n} b_{k,0}(i)^{v_k^j} = \prod_{k=1}^{d} b_{k,0}(i)^{v_k^j} = \prod_{k=1}^{d}(\frac{\tilde{z}_k}{\tilde{z}_0})^{v_k^j},$$

with $\tilde{b}_{j,0}(i)$'s a priori depending on curves C_i, but for $1 \leq j \leq d$ due to the latter formula de facto not depending on i. Let $z'_j := \tilde{b}_{j,0}(i)$, $1 \leq j \leq d$. Then $f_{i,\tilde{\gamma}(i)}(\tilde{b}(i)) = \tilde{b}(i)^{\tilde{\beta}(i)}\tilde{f}_i(\tilde{b}_{1,0}(i),\ldots,\tilde{b}_{d,0}(i)) = \tilde{b}(i)^{\tilde{\beta}(i)}\tilde{f}_i(z'_1,\ldots,z'_d)$. It follows that $\tilde{f}_i(z'_1,\ldots,z'_d) = 0$.

Repeating the above arguments starting with the choice of a curve $C_i \subseteq V(f_i)$ for every i, $1 \leq i \leq n$, we construct $\tilde{\gamma}(1),\ldots,\tilde{\gamma}(n) \in \mathbb{Q}^n$, $\tilde{\beta}(1),\ldots,\tilde{\beta}(n) \in \mathbb{Z}^n$ and $\tilde{b}(1),\ldots,\tilde{b}(n) \in (\mathbb{C}^*)^n$ such that for every i,

(i) $Q = \operatorname{conv}(A_{1,\tilde{\gamma}(i)} + \cdots + A_{n,\tilde{\gamma}(i)}) \cap \mathbb{Z}^n$, i.e. Q is the 'face' of P corresponding to $\tilde{\gamma}(i)$ and $A_{j,\tilde{\gamma}(i)}$ are the 'faces' of A_j corresponding to $\tilde{\gamma}(i)$ for every j, $1 \leq j \leq n$,

(ii) first d-coordinates of $\tilde{b}(i)$ are $z'_1,\ldots,z'_d$ (in particular do not depend on i),

(iii) Laurent polynomials $\tilde{f}_i := x^{-\tilde{\beta}(i)}f_{i,\tilde{\gamma}(i)}$ depend only on coordinates $(x_1,\ldots,x_d)$ and $\tilde{f}_i(z'_1,\ldots,z'_d) = 0$.

Since a decomposition of a face of a Minkowski sum of polytopes as the Minkowski sum of faces of the summands is unique, it follows that for every choice of i, j, k,

$A_{i,\tilde{\gamma}(j)} = A_{i,\tilde{\gamma}(k)}$. Let $\gamma := \tilde{\gamma}(1)$ and $b := \tilde{b}(1)$. Then for each i, $f_{i,\gamma}(b) = \sum_{\beta\in A_{i,\gamma}} a_{i,\beta}b^{\beta} = \sum_{\beta\in A_{i,\tilde{\gamma}(i)}} a_{i,\beta}b^{\beta} = f_{i,\tilde{\gamma}(i)}(b) = b^{\tilde{\beta}(i)}\tilde{f}_i(b_1,\ldots,b_d) = b^{\tilde{\beta}(i)}\tilde{f}_i(z'_1,\ldots,\tilde{z}'_d) = 0$. In other words, assuming that $\phi_{\mathcal{P}}$ does not preserve $\{f_1,\ldots,f_n\}$ at ∞ over 0, we derived that $(**)$ is not satisfied, as required for the proof of the 'only if' implication.

Next we prove the 'if' implication. Assume $(**)$ is not satisfied. Replacing f by f_{ψ} for a suitable injective affine transformation ψ of $\mathbb{Z}^n$ we may assume w.l.o.g. that $A_i \subseteq (\mathbb{Z}_+)^n$ for every i (similarly to the argument in the beginning of the proof of the 'only if' implication). Let $a := (a_1,\ldots,a_n) \in V(f_{1,\alpha},\ldots,f_{n,\alpha})$.

Lemma A.2. *We claim that for every i, $1 \le i \le n$, there is a curve $C_i \subseteq V(f_i)$ with parametrization*

$$C_i(t) = (a_1t^{\alpha_1} + \text{ h.o.t.}\,,\ldots, a_nt^{\alpha_n} + \text{ h.o.t.}) \ .$$

Proof. Indeed, fix an i, $1 \le i \le n$, and let

$$\hat{f}_i(t, w_1,\ldots,w_n) := t^{-d_i} f_i(t^{\alpha_1}(a_1 + w_1),\ldots,t^{\alpha_n}(a_n + w_n)),$$

where $t, w_1,\ldots,w_n$ are indeterminates and $d_i := \langle\alpha,\beta\rangle$ for $\beta \in A_{i,\alpha}$. Then $\hat{f}_i$ is a polynomial in $t, w_1,\ldots,w_n$, with a *vanishing* constant term, and therefore the origin O belongs to the hypersurface $V(\hat{f}_i)$ of $\mathbb{C}^{n+1}$. Since every point of a complex hypersurface (of a variety of dimension larger than or equal to two) is the origin of a germ of a complex analytic curve contained in the hypersurface, it follows that there are convergent Puiseux series $w_j(t)$, $1 \le j \le n$, in t with positive rational exponents and common denominators in $\mathbb{Z}_+$ such that

$$\hat{C}_i(t) := (t, w_1(t),\ldots,w_n(t))$$

is the parametrization of a curve contained in $V(\hat{f}_i)$ and centered at O. Then

$$C_i(t) := (t^{\alpha_1}(a_1 + w_1(t)),\ldots,t^{\alpha_n}(a_n + w_n(t))) \ ,$$

as claimed by the assertion of the lemma. □

Returning to the proof of the 'if' implication of theorem A.1 with the point a in the set $V(f_{1,\alpha}, \ldots, f_{n,\alpha})$ picked preceding lemma A.2 and with $C_i(t)$ from lemma A.2 it follows that $\phi_{\mathcal{P}}(C_i(t)) = [(a^{\alpha^0} t^{\langle \alpha, \alpha^0 \rangle} + \text{ h.o.t.}) : \cdots : (a^{\alpha^N} t^{\langle \alpha, \alpha^N \rangle} + \text{ h.o.t})]$. (The 'h.o.t.'s appear as they do because all $\alpha^j \in (\mathbb{Z}_+)^n$, $1 \leq j \leq N$). Multiplying every coordinate by t^{-d}, where $d := \min\{\langle \alpha, \alpha^j \rangle\}_{j=0}^N$, it follows that $\lim_{t \to 0} \phi_{\mathcal{P}}(C_i(t)) = b := [b_0 : \cdots : b_N]$, where

$$b_j = \begin{cases} a^{\alpha^j} & \text{if } \langle \alpha, \alpha^j \rangle = d, \quad 0 \leq j \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

(In particular, point b does not depend on the choices of curves C_i, $1 \leq i \leq n$, of lemma A.2.) Then $b \in \overline{C}_i \subseteq \overline{V(f_i)}$ for each i, and therefore $\phi_{\mathcal{P}}$ does not preserve $\{f_1, \ldots, f_n\}$ at ∞ over 0, which completes the proof of the theorem. $\square$

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