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How fast do polynomials grow on semialgebraic sets?

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ABSTRACT

We study the growth of polynomials on semialgebraic sets. For this purpose we associate a graded algebra with the set, and address all kinds of questions about finite generation. We show that for a certain class of sets, the algebra is finitely generated. This implies that the total degree of a polynomial determines its growth on the set, at least modulo bounded polynomials. We, however, also provide several counterexamples, where there is no connection between total degree and growth. In the plane, we give a complete answer to our questions for certain simple sets, and we provide a systematic construction for examples and counterexamples. Some of our counterexamples are of particular interest for the study of moment problems, since none of the existing methods seems to be able to decide the problem there. We finally also provide new three-dimensional sets, for which the algebra of bounded polynomials is not finitely generated.

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Contents

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1. Introduction

Let $\mathbb{R}[x]$ be the polynomial algebra in n variables $x = (x_1, \ldots, x_n)$. For $d \in \mathbb{N}$ we denote by $\mathbb{R}[x]_d = \{p \in \mathbb{R}[x] \mid \deg(p) \leq d\}$ the finite dimensional subspace of polynomials of total degree at most d. For a set $S \subseteq \mathbb{R}^n$ let

$$\mathcal{B}_d(S) := \left\{p \in \mathbb{R}[x] \;\middle|\; p^2 \leq q \text{ on } S, \text{ for some } q \in \mathbb{R}[x]_{2d}\right\}$$

denote the set of those polynomials, that grow on S as if they were of degree at most d. Clearly $\mathbb{R}[x]_d \subseteq \mathcal{B}_d(S)$ for all d, and $\mathcal{B}_d(S)$ is closed under addition, which follows for example from the inequality

$$\left(p + p'\right)^2 \leq \left(p + p'\right)^2 + \left(p - p'\right)^2 = 2p^2 + 2p'^2.$$

$\mathcal{B}_0(S)$ is the algebra of *bounded polynomials* on S, and each $\mathcal{B}_d(S)$ carries the structure of a $\mathcal{B}_0(S)$-module. More generally, $\mathcal{B}_d(S) \cdot \mathcal{B}_{d'}(S) \subseteq \mathcal{B}_{d+d'}(S)$, so we have a graded algebra

$$\mathcal{B}(S) := \bigoplus_{d \geq 0} \mathcal{B}_d(S).$$

$\mathcal{B}(S)$ can be identified with a subalgebra of $\mathbb{R}[x, t]$, where t is a single new variable, by identifying $p \in \mathcal{B}_d(S)$ with $p \cdot t^d$. Also note that

$$\mathcal{B}_d(S_1 \cup S_2) = \mathcal{B}_d(S_1) \cap \mathcal{B}_d(S_2) \quad \text{and} \quad \mathcal{B}(S_1 \cup S_2) = \mathcal{B}(S_1) \cap \mathcal{B}(S_2)$$

holds for $S_1, S_2 \subseteq \mathbb{R}^n$. This follows from the fact that q in the definition of $\mathcal{B}_d(S)$ can always be assumed to be globally non-negative. In fact, one can always take some $C + D\|x\|^{2d}$. Finally note that $\mathcal{B}_d(S)$ and thus $\mathcal{B}(S)$ do only depend on the behaviour of S at infinity; if S is changed inside of a compact set, no changes in $\mathcal{B}_d(S)$ and $\mathcal{B}(S)$ occur. In this paper, we consider the following questions:

Question 1.1. Is $\mathcal{B}(S)$ a finitely generated algebra?

Question 1.2. Is $\mathcal{B}_0(S)$ a finitely generated algebra?

Question 1.3. Is every $\mathcal{B}_d(S)$ a finitely generated $\mathcal{B}_0(S)$-module?

Question 1.4. If $\mathcal{B}_0(S) = \mathbb{R}$, is every $\mathcal{B}_d(S)$ a finite dimensional vector space?

Note that a positive answer to Question 1.1 yields a positive answer to all the other questions. Also note that Question 1.4 is just a special case of Question 1.3. Let us start with some examples.

Example 1.5. (1) Assume $S \subseteq \mathbb{R}^n$ contains a full-dimensional convex cone K (e.g. $S = \mathbb{R}^n$ or $S = [0,\infty)^n$). For any $0 \neq p \in \mathbb{R}[x]$ there is a point $0 \neq a \in K$, on which the highest degree part of p does not vanish. So, on the half-ray through a, the polynomial p^2 cannot be bounded by a polynomial of degree smaller than $2 \cdot \deg(p)$. This proves $\mathcal{B}_d(S) = \mathbb{R}[x]_d$, and the answer to all above questions is positive. Note that $\mathcal{B}(S)$ is generated by $t, x_1t, \dots, x_nt$.

(2) Let $S = ([0,1] \times \mathbb{R}) \cup (\mathbb{R} \times [0,1]) \subseteq \mathbb{R}^2$ be the union of a vertical and a horizontal strip. A polynomial p belongs to $\mathcal{B}_d(S)$ if and only if the degree of p as both a polynomial in x_1 and in x_2 is at most d. From this it is easy to see that the answer to all above questions is positive. Note that $\mathcal{B}_d(S) = \mathbb{R}[x]_d$ is not true here. Namely $x_1x_2 \in \mathcal{B}_1(S)$, since $x_1^2x_2^2 \leq x_1^2 + x_2^2$ on S. However, $\mathcal{B}(S)$ is generated by t, x_1t, x_1x_2t and x_2t.

(3) If S is bounded, then $\mathcal{B}_0(S) = \mathcal{B}_d(S) = \mathbb{R}[x]$ for all d, and the answer to Question 1.1 is obviously yes. In fact, $\mathcal{B}(S) = \mathbb{R}[x,t]$ here.

(4) The set S should be semialgebraic, if a positive answer to the above questions is to be expected. Indeed, consider $S = \{(a,b) \in \mathbb{R}^2 \mid 0 \leq a, \exp(a) \leq b\}$. Then for $p_d := \sum_{i=0}^d \frac{1}{i!}x_1^i$ and $(a,b) \in S$ we have

$$0 \leq p_d(a,b) \leq \exp(a) \leq q(a,b),$$

with $q = x_2$. So, $p_d \in \mathcal{B}_1(S)$ for all $d \in \mathbb{N}$. On the other hand, $\mathcal{B}_0(S) = \mathbb{R}$ is easily checked, so Question 1.4 has a negative answer.

(5) Even in the semialgebraic case, the answer to Question 1.4 can be negative. This example is Example 4.2 from [2]. Consider $S = \{(a,b) \in \mathbb{R}^2 \mid b^2(1-a^2) \geq 0\}$. So S is the union of a vertical strip and the x_1-axis. One checks that $\mathcal{B}_0(S) = \mathbb{R}$ holds. On the other hand, $(x_1^dx_2)^2 \leq x_2^2$ on S, so $x_1^dx_2 \in \mathcal{B}_1(S)$ for all d. So $\mathcal{B}_1(S)$ is not finite dimensional, and Question 1.4 has a negative answer. In particular, $\mathcal{B}(S)$ is not finitely generated. This example is however somewhat pathological, since S has a lower-dimensional part, i.e. is not regular.

(6) Another pathology arising from non-regular sets is the following. Let S be the x_1-axis in $\mathbb{R}^2$ alone. Then $\mathcal{B}_0(S)$ is not a finitely generated algebra, as one easily checks. This general phenomenon has already been observed in [17]. So also the answer to Question 1.1 is negative. On the other hand, each $\mathcal{B}_d(S)$ is a finitely generated $\mathcal{B}_0(S)$-module, generated by $1, x_1, \dots, x_1^d$.

We see that we should restrict to regular semialgebraic sets from now on, i.e. sets that are closures of open sets. Let us recall what is actually known concerning the above questions. It seems like Question 1.1 has not been explicitly studied for semialgebraic sets yet. The paper [17] deals with Question 1.2 and shows that the answer is yes in the two-dimensional regular case, and false in general for regular semialgebraic subsets of certain affine varieties in higher dimensions. The paper [5] proves that Question 1.2 already has a negative answer for regular semialgebraic subsets of $\mathbb{R}^3$, and an explicit three-dimensional such counterexample is provided. The example is based on a counterexample to Hilbert's 14th Problem of Kuroda. In the context of *moment problems*, Question 1.4 has been extensively studied. The works [6,9,14,18,24] all give positive answers for large classes of sets. To avoid confusion, we note that the question that is answered in these papers is, in fact, the following:

Question 1.6. If $\mathcal{B}_0(S) = \mathbb{R}$, does there exist a function $\psi : \mathbb{N} \to \mathbb{N}$, such that for any two polynomials $p, q \in \mathbb{R}[x]$ with $p, q \geq 0$ on S, one has

$$\deg(p) \leq \psi\big(\deg(p+q)\big)?$$

Lemma 1.7. *For any set $S \subseteq \mathbb{R}^n$, Question 1.4 and Question 1.6 are equivalent.*

Proof. First assume that each $\mathcal{B}_d(S)$ is a finite dimensional vector space, and let $\psi(d)$ be the maximal degree of a polynomial in $\mathcal{B}_d(S)$. If p, q are non-negative on S, then $p \leq p + q$ on S, so $p \in \mathcal{B}_{\deg(p+q)}(S)$. This shows $\deg(p) \leq \psi(\deg(p + q))$. Let conversely such a function ψ be given. Obviously ψ can be assumed to be non-decreasing. If $p^2 \leq q$ for some $q \in \mathbb{R}[x]_{2d}$, then $q - p^2$ and p^2 are non-negative on S. We obtain $\deg(p^2) \leq \psi(\deg(p^2 + (q - p^2))) \leq \psi(2d)$. This shows that $\mathcal{B}_d(S)$ is finite dimensional. □

Let us briefly recall some facts about the moment problem. Given a linear functional $\varphi : \mathbb{R}[x] \to \mathbb{R}$, one wants to determine whether it has a representing measure μ, i.e. whether

$$\varphi(p) = \int\limits_{\mathbb{R}^n} p \, d\mu$$

holds for all $p \in \mathbb{R}[x]$. Also the support of μ is of interest here. A result by Haviland [4] states that φ has a representing measure supported on a set $S \subseteq \mathbb{R}^n$ if and only if $\varphi(p) \geq 0$ holds for all polynomials p which are non-negative as functions on S. Unfortunately, describing all non-negative polynomials on S is a hard problem. So Haviland's theorem becomes particularly helpful if the non-negativity condition can be weakened. Towards this goal one restricts to *basic closed semialgebraic sets*

$$S = \big\{a \in \mathbb{R}^n \;\big|\; p_1(a) \geq 0, \ldots, p_r(a) \geq 0\big\},$$

where $p_1, \ldots, p_r \in \mathbb{R}[x]$. The polynomials which are obviously non-negative on S are of the form q^2 and $q^2 p_i$ for some $q \in \mathbb{R}[x]$.

We say that $p_1, \ldots, p_r$ *solve the moment problem for S* if the conditions

$$\varphi\left(q^2 p_i\right) \geq 0 \quad \forall q \in \mathbb{R}[x],\ i = 0, \ldots, r$$

(where $p_0 = 1$) are enough to ensure the existence of a representing measure for φ on S. Such a weakened positivity condition on φ can then for example be checked by a series of semidefinite programs (see for example [9]). The celebrated result of Schmüdgen [20] implies that if S is bounded, then the finitely many products $p_1^{e_1} \cdots p_r^{e_r}$ $(e_i \in \{0, 1\})$ always solve the moment problem for S. Another result of Schmüdgen [21] (see also [9,15]) provides a method to reduce the dimension in the moment problem. Given a nontrivial bounded polynomial $p \in \mathcal{B}_0(S)$, it is enough to check the moment problem on all fibres $S \cap \{p = \lambda\}$ of p in S. Since the problem is usually easier in lower dimensions, this is very helpful. Now the significance of Question 1.6 for the moment problem is the following. If $\mathcal{B}_0(S) = \mathbb{R}$ and Question 1.6 has a positive answer, then the moment problem is not solvable, at least in dimension ≥ 2, by a result of Scheiderer [19]. So it doesn't matter that the reduction result of Schmüdgen cannot be applied, since the problem is not solvable anyway. Since Question 1.6 has a positive answer in so many cases, it has been asked whether the answer is always yes, for *regular* semialgebraic sets, i.e. sets that are closures of open sets (see for example [16]). This would mean there is no gap between the results of Schmüdgen and Scheiderer. In this paper we show, among other things, that this is false. In particular, deciding the moment problem for our counterexamples seems to call for completely new methods.

Our contribution is the following. In Section 2 we show that Question 1.1 has a positive answer for a large class of sets in arbitrary dimension, built of so-called standard tentacles. In Section 3 we provide a first regular two-dimensional example, for which Question 1.4 (and thus any other of the questions as well) has a negative answer. We give a completely elementary and constructive proof. In Section 4 we use more elaborate techniques to examine planar sets. For certain simple sets (namely sets with a single 'tentacle'), we give complete answers to our Questions 1.1–1.4, and a partial solution to the moment problem. We provide a systematic way to produce more examples and counterexamples to our questions. The results show that in principle anything can happen, even for regular sets in the plane. The methods are based on the study of key forms for semidegree functions and corresponding (algebraic) compactifications of $\mathbb{C}^2$, mostly from [10,11,13]. We also remark that it is possible to extend the analysis in Sections 4.2 and 4.3 of the 'subdegrees' associated with planar sets to higher dimensional subalgebraic sets. It is however more technical in nature and will be part of a future work.

Let us finally remark how Question 1.2 relates to Question 1.1. In fact, we can always interpret the algebra $\mathcal{B}(S)$ as the algebra $\mathcal{B}_0(S')$ of another set S'. In this way, we can produce examples and counterexamples to Question 1.2 from examples and counterexamples to Question 1.1:

Proposition 1.8. *Let $S \subseteq \mathbb{R}^n$ be a set. Define*

$$S' := \left\{(a, s) \in \mathbb{R}^{n+1} \;\middle|\; a \in S, \|a\| \geq 1, \|a\|^2 s^2 \leq 1\right\}.$$

Then $\mathcal{B}(S) \cong \mathcal{B}_0(S')$ (via the identification $p \in \mathcal{B}_d(S) \leftrightarrow p \cdot t^d$, where t is the last coordinate function on $\mathbb{R}^{n+1}$).

Proof. We can assume that S contains only points with $\|a\| \geq 1$. For "$\subseteq$" we start with $p \in \mathcal{B}_d(S)$. This implies $|p| \leq D\|x\|^d$ on S, for some large enough D. So for $(a,s) \in S'$ we have

$$\left|p(a)s^d\right| \leq \left|p(a)\right| \frac{1}{\|a\|^d} \leq D.$$

So $p \cdot t^d \in \mathcal{B}_0(S')$, which proves the claim. For "$\supseteq$" take $q = \sum_{i=0}^d p_i(x)t^i \in \mathcal{B}_0(S')$. There is some C such that

$$\left|\sum_{i=0}^d \frac{p_i(a)}{\|a\|^i} r^i\right| = \left|q\left(a, \frac{r}{\|a\|}\right)\right| \leq C$$

for all $a \in S, r \in [0,1]$. From Lemma 1.9 below it follows that there is some D such that

$$\left|\frac{p_i(a)}{\|a\|^i}\right| \leq D$$

for all $a \in S$, and this implies $p_i \in \mathcal{B}_i(S)$, and thus $q \in \mathcal{B}(S)$. □

Lemma 1.9. *If a univariate polynomial $p \in \mathbb{R}[t]$ fulfills $|p| \leq C$ on $[0,1]$, then the size of the coefficients of p is bounded by a constant depending only on $\deg(p)$ and C.*

Proof. Write $p = \sum_{i=0}^d p_i t^i$. We have

$$-C \leq \sum_i p_i \left(\frac{1}{n}\right)^i \leq C$$

for all $n \in \mathbb{N}$. So the coefficient tuple $(p_0, \ldots, p_d)$ of p lies in a polytope whose normal vectors are $\pm(1, \frac{1}{n}, \frac{1}{n^2}, \ldots, \frac{1}{n^d})$, for $n = 1, \ldots, d+1$. Since these vectors are linearly independent, this polytope is compact. □

So, from examples where $\mathcal{B}(S)$ is not finitely generated, we obtain new examples for which $\mathcal{B}_0(S')$ is not finitely generated. For example, we get new regular examples in dimension three from the examples in Section 3 and Section 4.

2. Standard tentacles

In this section we prove that Question 1.1, the strongest of the above questions, has a positive answer for a large class of sets. We recall the definition of a *standard tentacle* from [14].

Definition 2.1. For $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$, a *standard z-tentacle* is a set

$$\left\{\lambda^z b := \left(\lambda^{z_1} b_1, \dots, \lambda^{z_n} b_n\right) \;\middle|\; \lambda \geq 1, b \in B\right\}$$

where $B \subseteq (\mathbb{R} \setminus \{0\})^n$ is a compact semialgebraic set with non-empty interior. We call B a *base* of the tentacle.

Any $z \in \mathbb{Z}^n$ defines a weighted degree $\deg_z$ on $\mathbb{R}[x]$, by assigning degree z_i to the variable x_i. We set

$$\varphi_z := \max\{0, z_1, \dots, z_n\}$$

and note that $\max\{\deg_z(p) \mid p \in \mathbb{R}[x]_d\} = d \cdot \varphi_z$.

For finite unions of standard tentacles, the modules $\mathcal{B}_d(S)$ from the last section can be described via these weighted degrees. For this let $z^{(1)}, \dots, z^{(m)} \in \mathbb{Z}^n$ be given. We write $\deg_i$ instead of $\deg_{z^{(i)}}$ and φ_i instead of $\varphi_{z^{(i)}}$.

Proposition 2.2. *Assume $S \subseteq \mathbb{R}^n$ is a finite union of standard tentacles, corresponding to the directions $z^{(1)}, \dots, z^{(m)} \in \mathbb{Z}^n$. Then for all $d \in \mathbb{N}$*

$$\mathcal{B}_d(S) = \left\{p \in \mathbb{R}[x] \;\middle|\; \deg_i(p) \leq d \cdot \varphi_i \textit{ for } i = 1, \dots, m\right\}.$$

In particular, $\mathcal{B}_d(S)$ is spanned as a vector space by the monomials contained in it.

Proof. "$\subseteq$": Let $p \in \mathcal{B}_d(S)$ with $p^2 \leq q$ on S, for some $q \in \mathbb{R}[x]_{2d}$. Since $q(\lambda^{z^{(i)}} b)$ is of degree at most $2d \cdot \varphi_i$ in λ, it follows that $\deg_i(p) \leq d \cdot \varphi_i$. At this point we need that tentacles have non-empty interior; there is some curve $\lambda^{z^{(i)}} b$ in the tentacle, on which p grows with $\deg_i(p)$. "$\supseteq$": Since $\mathcal{B}_d(S)$ is a vector space, we can assume that $p = x^\beta$ is a monomial. We can also assume $m = 1$. We have for $\lambda \geq 1$

$$p^2\left(\lambda^{z^{(1)}} b\right) = b^{2\beta} \lambda^{2z^{(1)}\beta^t} \leq C\lambda^{2\varphi_1 \cdot d},$$

for all b in the base B of the tentacle. Choose α with $|\alpha| \leq d$ and $z^{(1)}\alpha^t = \varphi_1 \cdot d$. Then choose $D > 0$ such that $C \leq D^2 b^{2\alpha}$ for all $b \in B$. We obtain $p^2 \leq (Dx^\alpha)^2$ on S, and thus $p \in \mathcal{B}_d(S)$. □

In [14, Theorem 5.4] it was shown that Question 1.6 has a positive answer, if S is a finite union of standard tentacles. We improve upon this now, while also simplifying the proof significantly:

Theorem 2.3. *If S is a finite union of standard tentacles, then $\mathcal{B}(S)$ is finitely generated.*

Proof. We use the same notation as before. Consider the set

$$M := \{(\alpha, d) \in \mathbb{N}^{n+1} \mid z^{(i)}\alpha^t \le d \cdot \varphi_i \text{ for } i = 1, \ldots, m\}.$$

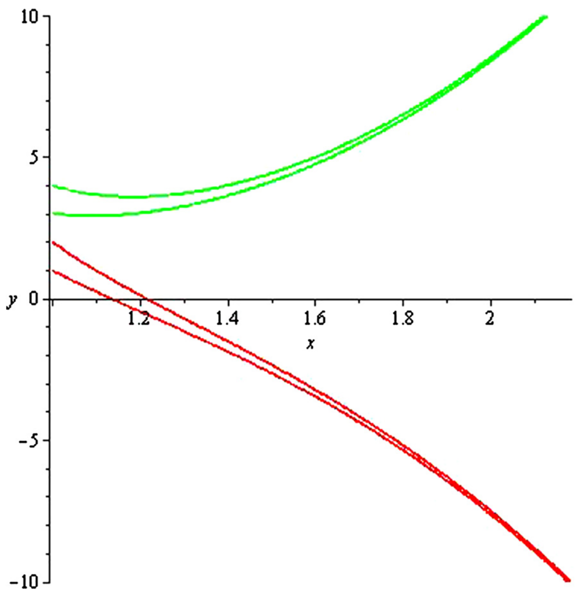
By Proposition 2.2, a polynomial from $\mathbb{R}[x, t]$ belongs to $\mathcal{B}(S)$ if and only if all its monomials do, and a monomial $x^\alpha t^d$ belongs to $\mathcal{B}(S)$ if and only if $(\alpha, d) \in M$. The lattice points in a rational convex cone form a finitely generated semigroup, by a well-known result of Hilbert. So if $(\alpha_1, d_1), \ldots, (\alpha_r, d_r)$ generate M, then the monomials $x^{\alpha_i} t^{d_i}$ generate $\mathcal{B}(S)$. □

3. A first counterexample

In this section we construct a first *regular* semialgebraic set S, for which Question 1.4, and thus all the other questions as well, have a negative answer. We will see more examples later, but for this one we give a completely elementary and constructive proof. We consider two sets

$$S_1 = \{(a, b) \in \mathbb{R}^2 \mid 1 \le a, 1 \le a^3 b + a^6 - a \le 2\}$$
$$S_2 = \{(a, b) \in \mathbb{R}^2 \mid 1 \le a, 1 \le a^3 b - a^6 - a \le 2\}$$

and set $S := S_1 \cup S_2$. Note that S is basic closed semialgebraic, i.e. definable by finitely many simultaneous polynomial inequalities. In fact, if $p = x^3 y + x^6 - x$ and $q = x^3 y - x^6 - x$, then $S = \mathcal{S}(x - 1, -(2 - p)(p - 1)(2 - q)(q - 1))$.



Theorem 3.1. *In the above example we have $\mathcal{B}_0(S) = \mathbb{R}$, but already $\mathcal{B}_1(S)$ is of infinite dimension.*

Proof. First note that S_1 consists precisely of the points $(\lambda, r\lambda^{-3} - \lambda^3 + \lambda^{-2})$ for $\lambda \geq 1$ and $r \in [1, 2]$. So if for $p \in \mathbb{R}[x_1, x_2]$ the λ-degree of $p(\lambda, r\lambda^{-3} - \lambda^3 + \lambda^{-2})$ is at most d, then p belongs to $\mathcal{B}_d(S_1)$. In fact, p^2 can then be bounded by some $C + Dx_1^{2d}$ on S_1. Note that $\mathcal{B}_0(S_1)$ consists precisely of those p where this λ-degree is ≤ 0.

If $q(x, y) := p(x, y - x^3 + x^{-2})$, then $p(\lambda, r\lambda^{-3} - \lambda^3 + \lambda^{-2}) = q(\lambda, r\lambda^{-3})$. So the λ-degree of $p(\lambda, r\lambda^{-3} - \lambda^3 + \lambda^{-2})$ equals $\deg_{(1,-3)}$ of the Laurent polynomial $p(x, y - x^3 + x^{-2})$. For S_2 the same is true with $+x^3$ instead of $-x^3$ everywhere. The claim will thus follow if we construct polynomials $p \in \mathbb{R}[x, y]$ of arbitrarily high degree, such that the $(1, -3)$-degree of both Laurent polynomials $p_+ = p(x, y + x^3 + x^{-2})$ and $p_- = p(x, y - x^3 + x^{-2})$ is at most 1, and if we show that degree ≤ 0 is only possible for constant polynomials p.

First consider $q = y^2 - x^6$. We find

$$q_\pm = q\big(x, y \pm x^3 + x^{-2}\big) = \pm 2x \pm 2x^3 y + x^{-4} + 2x^{-2} y + y^2.$$

Using this, we next consider $r = x^k y^l (y^2 - x^6)^m$ and the Laurent polynomials $r_\pm = r(x, y \pm x^3 + x^{-2})$. We find

$$\begin{aligned} r_\pm = &\sum_{a+b+c=l, d+e+f+g+h=m} \frac{l!m!}{a!b!c!d!e!f!g!h!} \\ &\times (\pm 1)^{b+d+e} 2^{d+e+g} x^{k+3b-2c+d+3e-4f-2g} y^{a+e+g+2h} \end{aligned}$$

From this formula we can read off the following facts:

- The coefficients of r_+ and r_- are the same up to signs. In fact, whether $b + d + e$ is even or odd only depends on the monomial $x^{k+3b-2c+d+3e-4f-2g} y^{a+e+g+2h}$ (in fact, only on $k + 3b - 2c + d + 3e - 4f - 2g$).
- The Newton polytope of $r_\pm$ has vertices

$$(-4m - 2l + k, 0),\ (k, 2m + l),\ (m + 3l + k, 0),\ (3m + 3l + k, m).$$

  There are monomials on the line from $(m + 3l + k, 0)$ to $(3m + 3l + k, m)$, and on parallel lines shifted by 5 to the left. No other monomials occur. This can be seen by checking that the $(1, -2)$-degree of the monomial $x^{k+3b-2c+d+3e-4f-2g} y^{a+e+g+2h}$ is

$$k - 2l + m + 5(b - f - g - h).$$

- The signs of the coefficients of r_- and r_+ obey the following rule. On the line through $(m+3l+k, 0)$ and $(3m+3l+k, m)$ the signs differ by $(-1)^{m+l}$. Going through parallel lines in steps of 5 to the left, the sign change oscillates from $+$ to $-$.

We are now ready to construct the desired polynomials. We start with

$$p^{(1)} = \big(y^2 - x^6\big)^m$$

for an arbitrarily large m. Many of the monomials of $p_+^{(1)}$ and $p_-^{(1)}$ have $(1,-3)$-degree ≤ 1 anyway. However, there are some which don't. If in the Newton polytope we follow the line from $(3m,m)$ in direction towards $(m,0)$, the first monomial $x^{3m}y^m$ is of degree 0, and the second monomial $x^{3m-2}y^{m-1}$ is of degree 1. We can tolerate both of them. The next one is however $x^{3m-4}y^{m-2}$, and here we have a $(1,-3)$-degree of 2. We now modify $p^{(1)}$ by adding

$$p^{(2)} = c \cdot x^2\left(y^2 - x^6\right)^{m-2}$$

to $p^{(1)}$, with a suitable coefficient c. Since $p_\pm^{(2)}$ gives rise to a Newton polytope with vertices

$$(-4m+10, 0),\ (2, 2m-4),\ (m, 0),\ (3m-4, m-2),$$

we can choose c to cancel the monomial $x^{3m-4}y^{m-2}$ in *both* $p_+^{(1)}$ and $p_-^{(1)}$ *at the same time*. This follows from the above sign considerations: the coefficients in $p_+^{(1)}$ and $p_-^{(1)}$ differ by $(-1)^m$ at this monomial, and the same is true for $p_+^{(2)}$ and $p_-^{(2)}$.

Now the new Laurent polynomials $(p^{(1)} + p^{(2)})_\pm$ both have one less of the bad monomials, namely $x^{3m-4}y^{m-2}$. At the same time, no new monomials arise. The coefficients have still the same modulus, and whether the sign changes is determined by the same rule as described above.

In this way one proceeds: Assume that all bad monomials up to $(3m-2(i-1), m-(i-1))$ have already been cancelled. Write $i = 3l + k$ with $0 \leq k \leq 4$ and $i - l$ even. Then a term

$$p^{(i)} = c \cdot x^k y^l \left(y^2 - x^6\right)^{m-i}$$

will allow to also cancel the monomial $(3m-2i, m-i)$ in both Laurent polynomials

$$\left(p^{(1)} + \cdots + p^{(i-1)}\right)_\pm$$

at the same time. This follows from the fact that $(-1)^{m-i+l} = (-1)^m$ since $i - l$ is even.

Once all bad monomials on this line are cancelled, one resumes with bad monomials on parallel lines to the left in a similar fashion. On the next line to the left, one for example has to write $i - 5 = 3l + k$, this time $i - l$ odd, and so on....

Now let us see why $\mathcal{B}_0(S) = \mathbb{R}$ holds. Assume that for some $p \in \mathbb{R}[x,y]$ we have $\deg_{(1,-3)} p_\pm \leq 0$. Since $\mathcal{B}_0(S)$ is an algebra, we can assume that the $(1,3)$-degree of p is even. One first checks that the highest degree part in the $(1,3)$-grading of p must then be $(y^2 - x^6)^m$ for some m, up to scaling. In fact, all monomials on the maximal $(1,3)$-line in the Newton polytope of $p_\pm$ that have $(1,-3)$-degree > 0 have to cancel. The corresponding linear equations are linearly independent, having Pascal matrices as coefficient matrices, as one easily computes. Since there are $2m+1$ possible coefficients in this highest degree part of p, and $2m$ equations, there is a unique solution up to scaling. And as we have seen before, $(y^2 - x^6)^m$ is such a solution.

Now the monomial $x^{3m-2}y^{m-1}$ in $(y^2 - x^6)^m_\pm$ has sign $(\pm 1)^m$, as we have seen. It can only cancel with terms coming from $x(y^2 - x^6)^{m-1}$ in p, by the above considerations. But here the sign will be $(\pm 1)^{m-1}$. So the cancellation cannot work for both substitutions at the same time. In other words, if p is not constant, then $p_\pm$ cannot both have $(1, -3)$-degree ≤ 0. This finishes the proof. □

Remark 3.2. Proposition 4.24 below gives an 'explanation' of the above counterexample in the language of Section 4.

As explained in the introduction, none of the existing methods to decide the moment problem seems to work for sets of this kind. The reduction result from [21] cannot be applied since $\mathcal{B}_0(S) = \mathbb{R}$, and since Question 1.6 has a negative answer, the usual way to see that the moment problem is unsolvable is also not successful. Sets of this kind seem to call for completely new methods.

4. Sets in the plane

In this section we use more elaborate techniques, mostly from [10,11,13], to examine planar sets in more detail. We will obtain many more examples and counterexamples to our question (see Example 4.23).

4.1. Degree like functions associated with a subset of $\mathbb{R}^n$

Let S be a subset of $\mathbb{R}^n$ and $\delta_S : \mathbb{R}[x_1, \ldots, x_n] \setminus \{0\} \to \mathbb{Z}$ be the function that maps f to the smallest d such that $f \in \mathcal{B}_d(S)$. Then δ_S is a *degree-like function* in the terminology of [13] (or equivalently, $-\delta_S$ is an *order function* in the terminology of [23]), i.e. δ_S satisfies properties (P1) and (P2) below:

(P1) $\delta_S(f + g) \le \max\{\delta_S(f), \delta_S(g)\}$ with equality if $\delta_S(f) \ne \delta_S(g)$, and
(P2) $\delta_S(fg) \le \delta_S(f) + \delta_S(g)$.

We note some (immediate) observations:

(P3) $\mathcal{B}(S) = \bigoplus_{d \ge 0}\{f : \delta_S(f) \le d\}$,
(P4) $\delta_S \le \deg$.

Now define $\bar{\delta}_S : \mathbb{R}[x_1, \ldots, x_n] \setminus \{0\} \to \mathbb{R}$ as

$$\bar{\delta}_S(f) := \lim_{n \to \infty} \delta_S\big(f^n\big)/n.$$

It is not hard to see that $\bar{\delta}_S$ is well defined and satisfies $\bar{\delta}_S(f^k) = k\bar{\delta}_S(f)$ for all f and k. We call $\bar{\delta}_S$ the *normalization* of δ_S. In the terminology of [23], $-\bar{\delta}_S$ is a *homogeneous order function*. We will examine the structure of $\bar{\delta}_S$ in more details for the case $n = 2$.

Definition 4.1. Let S be a semialgebraic subset of $\mathbb{R}^2$. For each $r > 0$, let B_r be the ball of radius r centered at the origin. For large enough r, the number of connected components of $S \setminus B_r$ becomes stable. Each of these components is called a *tentacle* of S.

4.2. The case of a single tentacle

Throughout this subsection we assume that S is a semialgebraic subset of $\mathbb{R}^2$ such that

(A1) S has only one tentacle, and
(A2) the tentacle of S is regular, i.e. is the closure of an open set.

Let $\bar{S}$ be the closure of S in $\mathbb{RP}^2$ and L_∞ be the line at infinity on $\mathbb{RP}^2$.

Lemma 4.2. *If $\bar{S}$ intersects L_∞ at more than one point, then $\delta_S = \bar{\delta}_S = \deg$.*

Proof. Since S has only one tentacle it follows that $\bar{S} \cap L_\infty$ is connected. It follows that $|\bar{S} \cap L_\infty| = \infty$. Choose coordinates $[X : Y : Z]$ on $\mathbb{RP}^2$ such that $(x, y) := (X/Z, Y/Z)$ are coordinates on $\mathbb{R}^2$. Choose an infinite sequence of points $P_i := [1 : c_i : 0] \in \bar{S} \cap L_\infty$ and curves $C_i \subseteq S$ such that $P_i \in \bar{C}_i$. Then $(1/x, y/x)$ are coordinates near each P_i and $\bar{C}_i$ has a Puiseux expansion at P_i of the form $y/x = c_i + \sum_{q\in\mathbb{Q},q>0} c_{i,q}(1/x)^q$, or equivalently, of the form $y = c_i x + \sum_{q\in\mathbb{Q},q<1} c_{i,q}x^q$. Now pick two polynomials $g_1, g_2 \in \mathbb{R}[x, y]$ with $d_1 := \deg(g_1) < d_2 := \deg(g_2)$. Then

$$g_i|_{C_j} = g_i(x, y)|_{y=c_j x+\sum_{q\in\mathbb{Q},q<1} c_{j,q}x^q} = g_{i,d_i}(1, c_j)x^{d_i} + \text{l.o.t.},$$

where g_{i,d_i} is the leading form of g_i and l.o.t. stands for ‘lower order terms’ (in x). If g_{2,d_2} is non-negative on $\mathbb{R}^2$, then it follows that for generic C_j, $g_2|_{C_j} > g_1|_{C_j}$ for sufficiently large $|x|$.

Now let $h \in \mathbb{R}[x, y]$, $d := \deg(h)$. Applying the preceding arguments with $g_2 := h^2$ and g_1 to be an arbitrary polynomial with $\deg(g_1) < 2d$ shows that $h \notin \mathcal{B}_e(S)$ for any $e < d$, i.e. $\delta_S \geq \deg$. Property (P4) then implies that $\delta_S = \deg$, as required. □

Now assume (A1) and (A2) hold and that $\bar{S}$ intersects L_∞ at only one point P. Choose a linear function u such that P is not on the closure of the line $u = 0$. Then (without changing δ_S), we may assume that for sufficiently large values of $|u|$, all points of S are bounded by curves $C_i := \{f_i(x, y) = 0\}$, $1 \leq i \leq 2$. Choose another linear function v such that (u, v) is linearly independent. Then $(1/u, v/u)$ is a set of coordinates (on $\mathbb{RP}^2$) near P and (the closure of) each C_i has a Puiseux expansion at P of the form

$v/u = \sum_{j\geq 0} a_{ij}(1/u)^{\tilde{\omega}_{ij}}$, with $0 \leq \tilde{\omega}_{i0} < \tilde{\omega}_{i1} < \cdots$ and $a_{ij} \in \mathbb{R}$, or $v = \phi_i(u)$, where $\phi_i(u) := \sum_{j\geq 0} a_{ij} u^{\omega_{ij}}$ where $\omega_{ij} := 1 - \tilde{\omega}_{ij}$.

Lemma 4.3. *Let ω be the largest (rational) number such that the coefficients of $u^{\omega'}$ in the expansion of C_i's are equal for all $\omega' > \omega$. Let $\phi(u)$ be the (common) part of ϕ_i's consisting of all terms with the exponent of u greater than ω. Let ξ be a new indeterminate and define $\delta^*_S : \mathbb{R}[x, y] \setminus \{0\} \to \mathbb{Q}$ as*

$$\delta^*_S(f) := \deg_u\bigl(f\bigl(u, \phi(u) + \xi u^{\omega}\bigr)\bigr). \tag{1}$$

Then

(1) $\bar{\delta}_S = \max\{0, \delta^*_S\}$.
(2) $\delta_S = \lceil \bar{\delta}_S \rceil$.
(3) *For each $f \in \mathbb{R}[x, y] \setminus \{0\}$, $f/|u|^{\delta^*_S(f)}$ is bounded outside a compact set on S.*

Proof. At first we claim that $\delta_S(u) = \bar{\delta}_S(u) = 1$. Indeed, since $\deg(u) = 1$, if the claim does not hold, then $\bar{\delta}_S(u) < 1$ and therefore there is a positive integer d and a polynomial $h \in \mathbb{R}[u, v] = \mathbb{R}[x, y]$ with $e := \deg(h) < 2d$ such that $u^{2d} < h$ on S. But it is impossible, since

$$h|_{C_i} = h(u, v)|_{v=\phi_i(u)} = cu^e + \text{l.o.t.}$$

for some $c \in \mathbb{R}$, and therefore $h|_{C_i} < u^{2d}|_{C_i}$ for large enough $|u|$. This proves the claim.

For each $t \in [0, 1]$, let $\phi_t(u) := t\phi_1(u) + (1 - t)\phi_2(u)$. Then for sufficiently large $|u|$, for each $t \in [0, 1]$, $v = \phi_t(u)$ defines a branch of real analytic curve C_t in S such that $\lim_{|u|\to\infty} C_t = P$. Now note that

$$\phi_t(u) = \phi(u) + \bigl(ta_1 + (1 - t)a_2\bigr)u^{\omega} + \text{l.o.t.} = \phi(u) + u^{\omega}\psi_t(u),$$

where a_i is the coefficient of u^{ω} in ϕ_i, and $\psi_t(u)$ is (a Puiseux series in $1/u$) of the form $\sum_{\omega'\leq 0} b_{\omega'}(t)u^{\omega'}$ with $b_0(t) = ta_1 + (1 - t)a_2$. Let $f \in \mathbb{R}[u, v]$ and $d := \deg_u(f(u, \phi(u) + \xi u^{\omega}))$. Then

$$f\bigl(u, \phi(u) + \xi u^{\omega}\bigr) = f_0(\xi)u^d + \text{l.o.t.}$$

where f_0 is a non-zero polynomial in ξ. It follows that

$$f|_{C_t} = f\bigl(u, \phi(u) + \xi u^{\omega}\bigr)\big|_{\xi=\psi_t(u)} = f_0\bigl(ta_1 + (1 - t)a_2\bigr)u^d + \text{l.o.t.}$$

We see that $f/|u|^d$ is bounded outside a compact set on S and consequently $\bar{\delta}_S(f) \leq \max\{0, d\}$ and $\delta_S(f) \leq \max\{0, \lceil d \rceil\}$.

Now we show that $\bar{\delta}_S(f) \geq \max\{0, d\}$ and $\delta_S(f) \geq \max\{0, \lceil d \rceil\}$. For this, w.l.o.g. we may assume that $d > 0$. Pick $h \in \mathbb{R}[u, v]$ with $\deg(h) < dk$ for some integer $k \geq 1$. Then $e := \deg_u(h(u, \phi(u) + \xi u^\omega)) \leq \deg(h) < dk$. Since

$$h|_{C_t} = h\big(u, \phi(u) + \xi u^\omega\big)\big|_{\xi=\psi_t(u)} = h_0\big(ta_1 + (1-t)a_2\big)u^e + \text{l.o.t.}$$

for some polynomial $h_0 \in \mathbb{R}[\xi]$, it follows that f^{2k} eventually becomes bigger on each C_t than h_0^2, so that $\delta_S(f_0^k) > \deg(h)$. Setting $k := 1$ immediately implies that $\delta_S(f) \geq \lceil d \rceil$, which proves the second assertion of the lemma. On the other hand, letting $k \to \infty$ and taking h so that $(dk - \deg(h))/k \to 0$, we see that $\bar{\delta}_S(f) = \lim_{k\to\infty} \delta_S(f_0^k)/k \geq d$, as required to prove the first assertion. The last assertion follows from the last sentence of the preceding paragraph. □

Definition 4.4. $\phi(u) + \xi u^\omega$ from identity (1) is called the *generic degree-wise Puiseux series* corresponding to S.

Remark 4.5. Note that δ_S^* is a *semidegree* (in the terminology of [13]), i.e. δ_S^* satisfies property (P2) of degree-like functions with an equality.

Corollary 4.6. *Assume (A1) and (A2) hold. Then $\bar{\delta}_S$ is a* semidegree *iff $\bar{\delta}_S = \bar{\delta}_S^*$ iff $\bar{\delta}_S^*$ is non-negative on $\mathbb{R}[x, y]$.*

Remark 4.7 *(A geometric interpretation of $\bar{\delta}_S$).* Assume (A1) and (A2) hold. With S we associate a compactification $\bar{X}_S$ of $X := \mathbb{R}^2$ and a curve $E_{S,\infty}$ 'at infinity' (i.e. an irreducible component of $\bar{X}_S \setminus X$) as follows. Let $\bar{X}_0 := \mathbb{RP}^2$ and $E_0 := \mathbb{RP}^2 \setminus \mathbb{R}^2$ be the line at infinity. If S satisfies the hypothesis of Lemma 4.2, i.e. $\bar{S}$ intersects E_0 at more than one point, then set $\bar{X}_S := \bar{X}_0$ and $E_{S,\infty} := E_0$. Otherwise, the two curves which bound the tentacle of S intersect E_0 at the same point O_0. Now assume $\bar{X}_i$ and E_i have been defined for some $i \geq 0$ and that the closure $\bar{S}$ of S in $\bar{X}_i$ intersects E_i at a single point O_i. Then define $\bar{X}_{i+1}$ to be the blow up of $\bar{X}_i$ at O_i and E_{i+1} to be the exceptional curve of the blow up. After finitely many steps we will arrive at $i \geq 1$ such that $\bar{S}$ intersects E_i at more than one point. Then set $\bar{X}_S := \bar{X}_i$ and $E_{S,\infty} := E_i$. Pick any linear function $u \in \mathbb{R}[x, y]$ such that $|u| \to \infty$ along the tentacle of S. Then Lemmas 4.2 and 4.3 state that

$$\bar{\delta}_S = \max\{0, \delta_S^*\}, \quad \text{where} \tag{2}$$

$$\delta_S^*(f) = \frac{\text{pole}_{E_{S,\infty}}(f)}{\text{pole}_{E_{S,\infty}}(u)} \quad \text{for all } f \in \mathbb{R}[x, y] \setminus \{0\}, \tag{3}$$

where $\text{pole}_{E_{S,\infty}}$ denotes the 'order of pole' along $E_{S,\infty}$. Now, identifying $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{C}^2$, we see that the complexification of $\bar{X}_S$ is a natural compactification of $\mathbb{C}^2$. This observation forms the basis for our applications in Sections 4.4.1 to 4.4.4 of the methods coming from compactifications of $\mathbb{C}^2$. Note that $\bar{X}_S$ is a compactification of $\mathbb{R}^2$ which is *compatible* with S in the sense of [17] (in fact, it is the 'minimal' S-compatible non-singular compactification of $\mathbb{R}^2$ which also 'dominates' $\mathbb{RP}^2$).

4.3. The general regular case

Proposition 4.8. *Let S be a semialgebraic subset of $\mathbb{R}^2$ such that every tentacle of S satisfies property (A2). Then*

$$\bar{\delta}_S = \max\{\bar{\delta}_T : T \text{ is a tentacle of } S\} = \max\bigl(\{0\} \cup \{\delta_T^* : T \text{ is a tentacle of } S\}\bigr). \quad (4)$$

Proof. It is clear that $\bar{\delta}_T \le \bar{\delta}_S$ for every tentacle T of S, which proves that LHS $\ge$ RHS in (4). The $\le$ direction follows from the last assertion of Lemma 4.3. □

Corollary 4.9. *Let S be as in Proposition 4.8. Then $\bar{\delta}_S$ is a* subdegree *(in the terminology of [13]), i.e. $\bar{\delta}_S$ is the maximum of finitely many semidegrees.*

Corollary 4.10. *Let S be as in Proposition 4.8. Then*

(1) *There exists a positive integer N such that $\bar{\delta}_S(f) \in \frac{1}{N}\mathbb{Z}$ for all $f \in \mathbb{R}[x, y] \setminus \{0\}$.*
(2) *Define*

$$\bar{\mathcal{B}}_d(S) := \{f \in \mathbb{R}[x, y] : \bar{\delta}_S(f) \le d\},$$
$$\bar{\mathcal{B}}(S) := \bigoplus_{d \ge 0} \bar{\mathcal{B}}_d(S).$$

Then $\mathcal{B}_d(S) = \bar{\mathcal{B}}_d(S)$ for all $d \ge 0$ and $\mathcal{B}(S) = \bar{\mathcal{B}}(S)$.

Proof. Assertion 1 follows immediately from the Lemma 4.3, Proposition 4.8 and the observation that a semialgebraic set has only finitely many tentacles. Assertion 2 follows from Proposition 4.8 and Assertion 2 of Lemma 4.3. □

4.4. Semidegrees on $\mathbb{C}[x, y]$ and corresponding compactifications of $\mathbb{C}^2$

4.4.1. Background

Let δ be a semidegree (see Remark 4.5) on $\mathbb{C}[x, y]$ defined as:

$$\delta(f) := \deg_x\bigl(f\bigl(x, \phi(x) + \xi x^{\omega}\bigr)\bigr), \quad (5)$$

where ξ is an indeterminate, $\phi \in \mathbb{C}[x^{1/N}, x^{-1/N}]$ for some positive integer N and $\omega \in \mathbb{Q}$, $\omega < \operatorname{ord}_x(\phi)$. Associated with δ there is a finite sequence of elements in $\mathbb{C}[x, x^{-1}, y]$ called the *key forms* of δ (see [10, Definition 3.17]). The sequence starts with $f_0 := x$, $f_1 := y$, and continues until there is an element in $\mathbb{C}[x, x^{-1}, y]$ which can be expressed as a polynomial in the computed key forms and whose δ-value is smaller than the 'expected' value. An algorithm and detailed example for the computation of key forms of δ from ϕ and ω appears in [10, Section 3.3].

Example 4.11.

$\phi(x) + \xi x^{\omega}$	Key forms
$\xi x^{p/q}$	x, y
$cx^{\frac{p}{q}} + \xi x^{\omega}$, p, q rel. prime integers, $q > 0$, $\omega < \frac{p}{q}$	$x, y, y^q - c^q x^p$
$x^{5/2} + x^{-3/2} + \xi x^{-5/2}$	x, y, $y^2 - x^5$, $y^2 - x^5 - 2x$
$x^{5/2} + x^{-1} + x^{-3/2} + \xi x^{-5/2}$	x, y, $y^2 - x^5$, $y^2 - x^5 - 2x^{-1}y$, $y^2 - x^5 - 2x^{-1}y - 2x$

Given a (normal) algebraic variety Y and a codimension one irreducible subvariety V of Y, the *order of pole* along V defines a semidegree on the field of rational functions on Y. Given a semidegree δ on $\mathbb{C}[x, y]$, the following proposition gives the construction of a compact algebraic variety containing $\mathbb{C}^2$ which 'realizes' δ (as the order of pole) along some curve.

Proposition 4.12. *(See [11, Proposition 2.10].) Let δ be defined as in (5). Assume that $\delta \neq$ deg. Pick the smallest positive integer N such that $N\delta$ is integer-valued. Then there exists a unique compactification $\bar{X}$ of $X := \mathbb{C}^2$ such that*

(1) *$\bar{X}$ is projective and normal.*
(2) *$\bar{X}_\infty := \bar{X} \setminus X$ has two irreducible components C_1, C_2.*
(3) *The semidegree on $\mathbb{C}[x, y]$ corresponding to C_1 and C_2 are respectively* deg *and $N\delta$.*

Moreover, all singularities of $\bar{X}$ are rational.

Theorem 4.13 *(Characterizing when δ is non-negative or positive). (See [11, Theorem 1.4].) Let δ be a semidegree on $\mathbb{C}[x, y]$ and let $g_0, \dots, g_{n+1}$ be the key forms of δ in (x, y)-coordinates. Then*

(1) *δ is non-negative on $\mathbb{C}[x, y] \setminus \mathbb{C}$ iff $\delta(g_{n+1})$ is non-negative.*
(2) *δ is positive on $\mathbb{C}[x, y] \setminus \mathbb{C}$ iff one of the following holds:*
   (a) *$\delta(g_{n+1})$ is positive,*
   (b) *$\delta(g_{n+1}) = 0$ and $g_k \notin \mathbb{C}[x, y]$ for some k, $0 \leq k \leq n + 1$, or*
   (b′) *$\delta(g_{n+1}) = 0$ and $g_{n+1} \notin \mathbb{C}[x, y]$.*

Moreover, conditions 2b *and* 2b′ *are equivalent.*

With a semidegree δ on $\mathbb{C}[x, y]$ we associate a graded ring

$$\mathbb{C}[x,y]^\delta := \bigoplus_{d\geq 0}\{f : \delta(f) \leq d\}. \tag{6}$$

In the case δ is realized (as in Proposition 4.12) as the order of pole along a curve on a normal surface, $\mathbb{C}[x,y]^\delta$ can be interpreted as the graded ring of global sections of a divisor. The following results exploit this connection to study finiteness properties of $\mathbb{C}[x,y]^\delta$.

Proposition 4.14. *Let f_δ be the last key form of δ. Assume δ is positive on $\mathbb{C}[x,y] \setminus \mathbb{C}$ and f_δ is* not *a polynomial. Then $\mathbb{C}[x,y]^\delta$ is* not *finitely generated over $\mathbb{C}$. Moreover,*

(1) *If $\delta(f_\delta) > 0$, then $\mathbb{C}[x,y]^\delta_d := \{f : \delta(f) \leq d\}$ is a finite dimensional vector space over $\mathbb{C}$ for all d.*
(2) *If $\delta(f_\delta) = 0$, then there exists $d > 0$ such that $\mathbb{C}[x,y]^\delta_d$ is an infinite dimensional vector space over $\mathbb{C}$.*

Proof. It follows from the assumptions that $\delta \neq \deg$. Let $\bar{X}$ be the compactification of $X := \mathbb{C}^2$ from Proposition 4.12. Since δ is positive on $\mathbb{C}[x,y] \setminus \mathbb{C}$, Theorem 4.13 implies that $\delta(f_\delta) \geq 0$. At first assume $\delta(f_\delta) > 0$. It then follows by the same arguments as in the proof of [10, Theorem 1.14] that $\mathbb{C}[x,y]^\delta$ is not finitely generated over $\mathbb{C}$ and that there exists a divisor D on $\bar{X}$ such that

$$\mathbb{C}[x,y]^\delta = \bigoplus_{d\geq 0} H^0\big(\bar{X}, \mathcal{O}_{\bar{X}}(dD)\big).$$

Since $H^0(\bar{X}, \mathcal{O}_{\bar{X}}(dD))$ is a finite dimensional vector space over $\mathbb{C}$ for each d, this proves the proposition for the case that $\delta(f_\delta) > 0$.

Now assume $\delta(f_\delta) = 0$. It then follows from [11, identity (11)] that $(C_1, C_1) = 0$. Since the singularities of $\bar{X}$ are rational, this implies that C_1 and C_2 are $\mathbb{Q}$-Cartier divisors. It follows that $D_k := kC_1 + C_2$ is a nef ($\mathbb{Q}$-Cartier) divisor on $\bar{X}$ for all $k \gg 0$. Pick any positive integer e such that eC_i is a Cartier divisor for each i. Then eD_k is an ample Cartier divisor on $\bar{X}$ for all $k \gg 0$. Let $\pi : \tilde{X} \to \bar{X}$ be a resolution of singularities of $\bar{X}$. Let H be a fixed ample divisor and $K_{\tilde{X}}$ be the canonical divisor on $\tilde{X}$. W.l.o.g. we may assume that the supports of both H and K_X are contained in $\tilde{X} \setminus X$. Since $H + \pi^*(eD_k)$ is also ample for each $k \gg 0$, it follows from a theorem of Reider (see e.g. [7]) that $\tilde{D}_k := 3H + \pi^*(3eD_k) + K_{\tilde{X}}$ is base-point free for each $k \gg 0$. Let c_1 (r esp. c_2) be the coefficient of C_1 (resp. C_2) in $3H + K_{\tilde{X}}$. Then for all $k \gg 0$, since $\tilde{D}_k$ is base-point free, it follows that there exists $f_k \in \mathbb{C}[x,y]$ such that $\deg(f_k) = c_1 + 3ek$ and $\delta(f_k) \leq c_2 + 3e$. Let $d := c_2 + 3e$. Consequently the degree d part of $\mathbb{C}[x,y]^\delta$ is infinite dimensional over $\mathbb{C}$, as required. □

Proposition 4.15. *Let f_δ be the last key form of δ. Assume one of the following conditions hold:*

(1) all *key forms of* δ *are polynomials (equivalently,* f_δ *is a polynomial), or*
(2) $\delta(f_\delta) < 0$ *(equivalently, there is* $f \in \mathbb{C}[x,y] \setminus \{0\}$ *such that* $\delta(f) < 0$*).*

Then $\mathbb{C}[x,y]^\delta$ *is finitely generated over* $\mathbb{C}$.

Proof. Let the key forms of δ be $f_0 = x, f_1 = y, f_2, \ldots, f_l$. At first we assume that all f_k's are polynomials. Pick a positive integer N such that $N\delta$ is integer-valued. It suffices to show that $\mathbb{C}[x,y]^{N\delta}$ is finitely generated over $\mathbb{C}$, where

$$\mathbb{C}[x,y]^{N\delta} = \bigoplus_{d \geq 0} \{f : N\delta(f) \leq d\},$$

(since $\mathbb{C}[x,y]^{N\delta}$ is integral over $\mathbb{C}[x,y]^\delta$). Let $e_j := N\delta(f_j)$, $1 \leq j \leq l$. Let us denote by $(f_j)_{e_j}$ the 'copy' of f_j in the e_j-th graded component of $\mathbb{C}[x,y]^{N\delta}$. We now show that $\mathbb{C}[x,y]^{N\delta}$ is generated as a $\mathbb{C}$-algebra by $\{(1)_1\} \cup \{(f_j)_{e_j} : 0 \leq j \leq l\}$. Indeed, pick $f \in \mathbb{C}[x,y]$. Recall (from [10, Proposition 3.28]) that for each $j \geq 1$, f_j is monic in y (as a polynomial in y with coefficients in $\mathbb{C}[x]$) and $\deg_y(f_j)$ divides $\deg_y(f_{j+1})$ for $1 \leq j \leq l-1$. It follows that given an $f \in \mathbb{C}[x,y]$, f has an expression of the form

$$f = \sum_{\alpha \in \mathbb{Z}^{l+2}_{\geq 0}} a_\alpha x^{\alpha_0} f_1^{\alpha_1} \cdots f_l^{\alpha_l},$$

for $a_\alpha \in \mathbb{C}$ and $\alpha_j < \deg_y(f_{j+1})/\deg_y(f_j)$ for $1 \leq j \leq l-1$. [8, Theorem 16.1] then implies that $N\delta(f) = \max\{N\delta(x^{\alpha_0} f_1^{\alpha_1} \cdots f_l^{\alpha_l}) : a_\alpha \neq 0\}$. It then immediately follows that

$$(f)_{N\delta(f)} = \sum_{\alpha \in \mathbb{Z}^{l+2}_{\geq 0}} a_\alpha \big((x)_{e_0}\big)^{\alpha_0} \big((f_1)_{e_1}\big)^{\alpha_1} \cdots \big((f_l)_{e_l}\big)^{\alpha_l} \big((1)_1\big)^{N\delta(f) - \sum \alpha_j e_j},$$

as required to show that $\mathbb{C}[x,y]^{N\delta}$ is finitely generated as an algebra over $\mathbb{C}$.

Now assume that $\delta(f_l) < 0$. W.l.o.g. we may (and will) also assume that f_l is *not* a polynomial. Define a map $\nu : \mathbb{C}[x,y] \setminus \{0\} \to \mathbb{Z}^2$ as follows: for every $g \in \mathbb{C}[x,y] \setminus \{0\}$, $g|_{y=\phi(x)+\xi x^\omega}$ is of the form $x^\alpha g_\alpha(\xi) + \text{l.o.t.}$ for some $\alpha \in \mathbb{Q}$ and $g_\alpha \in \mathbb{C}[\xi]$ (where l.o.t. denotes 'lower-order terms' in x). Then set $\nu(g) := (N\alpha, \deg_\xi(g_\alpha))$.

Claim 4.15.1. *There exists* $f \in \mathbb{C}[x,y]$ *such that* $\nu(f) = (0,k)$ *for some positive integer* k.

Proof. Indeed, [11, Theorem 1.7] implies that there exists $h \in \mathbb{C}[x,y]$ such that $\delta(h) < 0$. Then $f := h^a x^b$ for suitable non-negative integers a, b satisfies the claim. □

Let $G_+ := \{\nu(g) : g \in \mathbb{C}[x,y],\ \delta(g) \geq 0\}$. Then G_+ is a sub-semigroup of $\mathbb{Z}^2_{\geq 0}$. Moreover, since $\nu(x)$ is of the form $(k_1, 0)$ and $\nu(f)$ (where f is as in Claim 4.15.1) is of the form $(0, k_2)$ for positive integers k_1, k_2, it follows that $\mathbb{Z}^2_{\geq 0}$ is *integral* over G_+, and therefore G_+ is a finitely generated semigroup. Pick $g_1, \dots, g_k$ such that $\nu(g_j)$'s generate G_+. Let $d_j := \delta(g_j)$, $1 \le j \le k$. The proposition follows from the following claims. □

Claim 4.15.2. *$(g_j)_{d_j}$, $1 \le j \le k$, generate $\mathbb{C}[x,y]^\delta$ as an algebra over $\mathbb{C}[x,y]^\delta_0 := \{f : \delta(f) \le 0\}$.*

Proof. Let $\prec$ be the lexicographic order on $\mathbb{Z}^2_{\ge 0}$. The claim follows from the observation that if $\nu(g) = \sum \alpha_j \nu(g_j)$, then there exists (a unique) $c \in \mathbb{C} \setminus \{0\}$ such that $\nu(g - cg_1^{\alpha_1} \cdots g_k^{\alpha_k}) \prec \nu(g)$. □

Claim 4.15.3. *$\mathbb{C}[x,y]^\delta_0$ is a finitely generated $\mathbb{C}$-algebra.*

Proof. Note that $\delta \ne \deg$. Let $\bar{X}$ be the compactification of $\mathbb{C}^2$ from Proposition 4.12. Since the singularities of $\bar{X}$ are rational, it follows that C_1 is a $\mathbb{Q}$-Cartier divisor. Since $\delta(f_l) < 0$, [11, Proposition 2.10] implies that $(C_1, C_1) > 0$. Moreover, since f_l is not a polynomial, [11, Proposition 2.11] implies that every compact curve on $\bar{X}$ intersects C_1. It follows from the Nakai–Moishezon criterion that C_1 is an ample divisor and therefore $\bar{X} \setminus C_1$ is an affine variety. The claim follows from the observation that $\mathbb{C}[x,y]^\delta_0$ is precisely the ring of regular functions on $\bar{X} \setminus C_1$. □

Corollary 4.16 *(Characterization of finiteness properties of $\mathbb{C}[x,y]^\delta$). Let f_δ be the last key form of δ.*

(1) *$\mathbb{C}[x,y]^\delta$ is* not *a finitely generated algebra over $\mathbb{C}$ iff both of the following conditions hold:*
   (a) *δ is non-negative on $\mathbb{C}[x,y]$ (or equivalently, $\delta(f_\delta) \ge 0$), and*
   (b) *there is a key form of δ which is not a polynomial (or equivalently, f_δ is not a polynomial).*
(2) *There exists $d > 0$ such that $\mathbb{C}[x,y]^\delta_d$ is* not *a finite dimensional module over $\mathbb{C}[x,y]^\delta_0$ iff both of the following conditions hold:*
   (a) *$\delta(f_\delta) = 0$, and*
   (b) *there is a key form of δ which is not a polynomial (or equivalently, f_δ is not a polynomial).*
(3) *$\mathbb{C}[x,y]^\delta_0 = \mathbb{C}$ in the case that conditions* 1a *and* 1b *are satisfied.*

Proof. Assertions 1 and 3 follow immediately from Theorem 4.13 and Propositions 4.14 and 4.15. The ($\Leftarrow$) direction of Assertion 2 follows from Assertion 2 of Proposition 4.14; we now show the ($\Rightarrow$) direction. If $\delta(f_\delta) > 0$, then we are done by Assertion 1 of Proposition 4.14. So we may assume one of the following conditions hold:

(a) $\delta(f_\delta) < 0$, or
(b) $\delta(f_\delta) = 0$ and f_δ is a polynomial.

Define $\nu : \mathbb{C}[x,y] \setminus \{0\} \to \mathbb{Z}^2$ and the semigroup $G_+ \subseteq \mathbb{Z}^2_{\geq 0}$ as in the proof of Proposition 4.15. It follows by the same arguments as in the proof of Proposition 4.15 that G_+ is finitely generated semigroup (in case (b), the role of f of Claim 4.15.1 is played by f_δ). Pick $g_1, \dots, g_k$ such that $\nu(g_j)$'s generate G_+. Let $d_j := \delta(g_j)$, $1 \leq j \leq k$. Then the same arguments as in the proof of Claim 4.15.2 show that $(g_j)_{d_j}$, $1 \leq j \leq k$, generate $\mathbb{C}[x,y]^\delta$ as an algebra over $\mathbb{C}[x,y]^\delta_0$. Pick those g_j's such that $d_j > 0$, we may assume these are $g_1, \dots, g_l$, $l \leq k$. Then for each $d > 0$, $\mathbb{C}[x,y]^\delta_d$ is generated as a module over $\mathbb{C}[x,y]^\delta_0$ by $\{\prod_{j=1}^l g_j^{\alpha_j} : \alpha_j \in \mathbb{Z}^l_{\geq 0}, \sum_{j=1}^l \alpha_j d_j = d\}$. □

4.4.2. Applications to the single tentacle case

Throughout this subsection S is assumed to be a semialgebraic subset of $\mathbb{R}^2$ which satisfies (A1) and (A2). Propositions 4.17–4.19 completely answer Questions 1.1–1.4 for S. Proposition 4.20 partially solves the moment problem for such S.

Proposition 4.17 *(A necessary and sufficient criterion for $\bar{\delta}_S$ to be a semidegree). Let f_S be the last key form of δ^*_S. Then the following are equivalent:*

(1) *$\bar{\delta}_S$ is a semidegree.*
(2) *δ^*_S is non-negative on $\mathbb{R}[x,y]$.*
(3) *$\delta^*_S(f_S) \geq 0$.*

Proof. Recall that $\bar{\delta}_S = \max\{0, \delta^*_S\}$ (Lemma 4.3). Since δ^*_S is a semidegree and $\bar{\delta}_S \neq 0$, it follows that $\bar{\delta}_S$ is a semidegree iff $\bar{\delta}_S = \delta^*_S$ iff $\delta^*_S \geq 0$ on $\mathbb{R}[x,y]$. Finally, Theorem 4.13 implies that $\delta^*_S \geq 0$ on $\mathbb{R}[x,y]$ iff $\delta^*_S(f_S) \geq 0$, as required. □

Proposition 4.18 *(A necessary and sufficient criterion for $\mathcal{B}_0(S) = \mathbb{R}$). Let f_S be the last key form of δ^*_S. Then $\mathcal{B}_0(S) = \mathbb{R}$ iff one of the following holds:*

(1) *$\delta^*_S(f_S) > 0$, or*
(2) *$\delta^*_S(f_S) = 0$ and f_S is* not *a polynomial.*

Proof. This is an immediate corollary of Theorem 4.13. □

Proposition 4.19 *(Characterization of finiteness properties of $\mathbb{C}[x,y]^\delta$). Let f_S be the last key form of δ^*_S.*

(1) *$\mathcal{B}(S)$ is* not *a finitely generated algebra over $\mathcal{B}_0(S)$ iff both of the following conditions hold:*
   (a) *δ^*_S is non-negative on $\mathbb{R}[x,y]$ (or equivalently, $\delta^*_S(f_S) \geq 0$), and*
   (b) *a key form of δ^*_S is not a polynomial (or equivalently, f_S is not a polynomial).*
(2) *There exists $d > 0$ such that $\mathcal{B}_d(S)$ is* not *a finite dimensional module over $\mathcal{B}_0(S)$ iff both of the following conditions hold:*

(a) $\delta_S^*(f_S) = 0$, *and*
(b) *a key form of* δ_S^* *is not a polynomial (or equivalently,* f_S *is not a polynomial).*

Proof. It follows immediately from Corollary 4.16. □

Proposition 4.20 *(Partial solution of the moment problem for S).*

(1) *The moment problem for* S *cannot be solved by finitely many polynomials in the following cases:*
 (a) $\delta_S^*(f_S) > 0$, *or*
 (b) $\delta_S^*(f_S) = 0$, f_S *is a polynomial, and the curve* $f_S = \xi$ *for generic* ξ *has genus* ≥ 1.
(2) *Assume one of the following conditions is satisfied:*
 (a) $\delta_S^*(f_S) < 0$, *or*
 (b) $\delta_S^*(f_S) = 0$ *and* f_S *is a polynomial, and the curve* $f_S = \xi$ *for generic* ξ *is rational.*

 Then there is a compact subset V *of* S *such that the moment problem for* $S \setminus V$ *can be solved by finitely many polynomials.*

Remark 4.21. The only case not covered by Proposition 4.20 (for those S which satisfy (A1) and (A2)) occurs when $\delta_S^*(f_S) = 0$ and f_S is *not* a polynomial. As explained at the end of Section 3, 'sets of this kind seem to call for completely new methods.'

Proof of Proposition 4.20. Assertion 1a follows from Assertion 2 of Corollary 4.16 and the discussion following Lemma 1.7. Assertion 1b follows from [18, Corollary 3.10] coupled with the following observations:

(1) for all but finitely many ξ, the curve $C_\xi := \{f_S = \xi\}$ is smooth (by Bertini's theorem),
(2) C_ξ has only one point at infinity (by [10, Proposition 4.2]), and
(3) there exists a non-empty open interval $I \subseteq \mathbb{R}$ such that for all $\xi \in \mathbb{R}$, the point at infinity of C_ξ belongs to the closure (in $\mathbb{RP}^2$) of $S \cap C_\xi$.

For Assertion 2, choose linear coordinates (u, v) on $\mathbb{R}^2$ such that $u \to \infty$ along the tentacle of S and (u, v) satisfy the hypotheses of Lemma 4.3 (i.e. the point of intersection of the closure $\bar{S}$ of S in $\mathbb{RP}^2$ and the line at infinity is not on the closure of the line $u = 0$). At first assume we are in the situation of 2a. Then Theorem 4.13 implies that there is $f \in \mathbb{R}[u, v]$ such that $\delta_S^*(f) < 0$. Choose positive integers a, b such that $h := u^a f^b$ satisfies $\delta_S^*(h) = 0$. For each $\xi \in \mathbb{R}$, let C_ξ be the curve $h = \xi$ and for all $r \geq 0$, let $S_r := \{(u, v) \in S : u \geq r\}$. Then for sufficiently large r, we have that

(1) S_r is defined by $\{u \geq r, h_1 \geq 0, h_2 \geq 0\}$, where $h_1, h_2 \in \mathbb{R}[u, v]$ which 'define the boundaries of the tentacle of S_r',

(2) h is bounded on S_r,
(3) C_ξ has a real point at infinity (namely the point in the closure of the line $u = 0$) which is not in the closure (in $\mathbb{RP}^2$) of S_r,
(4) $C_\xi \cap S_r$ does not intersect $\{h_i = 0\}$ for $i \in \{1, 2\}$.

Then [21, Theorem 1] and [18, Theorem 3.11] imply that $\{u-r, h_1, h_2\}$ solves the moment problem on S_r.

Now assume the hypotheses of 2b are satisfied. Since C_ξ has only one point at infinity for all ξ (by [10, Proposition 4.2]), it follows from the Abhyankar–Moh–Suzuki theorem [1,22] that f_S is a *polynomial coordinate* on $\mathbb{C}^2$, i.e. there exists a polynomial $g \in \mathbb{C}[u, v]$ such that $\mathbb{C}[f_S, g] = \mathbb{C}[u, v]$. It is then not hard to show using Jung's theorem (see e.g. [3]) on polynomial automorphisms of the plane that f_S is, in fact, a polynomial coordinate on $\mathbb{R}^2$, i.e. there exists $g \in \mathbb{R}[u, v]$ such that $\mathbb{R}[f_S, g] = \mathbb{R}[u, v]$. W.l.o.g. we may assume that $g \to \infty$ along the tentacle of S. Let $S_r := \{(u, v) \in S : g(u, v) \geq r\}$. There exists $h_1, h_2 \in \mathbb{R}[u, v]$ such that $S_r = \{g \geq r, h_1 \geq 0, h_2 \geq 0\}$ for all sufficiently large r. [21, Theorem 1] and [6, Theorem 2.2] then imply that $\{g - r, h_1, h_2\}$ solves the moment problem on S_r for r large enough. This completes the proof of Assertion 2. □

4.4.3. Explicit construction of single tentacles

In this subsection we give an algorithm to construct tentacles S with desired behaviour of $\mathcal{B}(S)$. The algorithm consists of the construction of a sequence of elements $f_0, \ldots, f_l$, $l \geq 1$, in $\mathbb{R}[x, x^{-1}, y]$ which would be the key forms of the corresponding semidegree δ_S^*. It is a straightforward adaptation of Maclane's construction of key polynomials in [8].

Initial step: Set $f_0 := x$ and $f_1 := y$. Pick $\omega_1 \in \mathbb{Q}$ and set $\omega_0 := 1$.

Inductive step: Assume f_j's and ω_j's have been constructed up to some $k \geq 1$. Let p_k be the *smallest* positive integer such that $p_k\omega_k$ is in the additive group generated by $\omega_0, \ldots, \omega_{k-1}$. Then $p_k\omega_k$ can be *uniquely* expressed in the form

$$p_k\omega_k = \alpha_{k,0}\omega_0 + \alpha_{k,1}\omega_1 + \cdots + \alpha_{k,k-1}\omega_{k-1}$$

where $\alpha_{k,j}$'s are integers such that $0 \leq \alpha_{k,j} < p_j$ for all $j \geq 1$ (note that there is no restriction on the range of $\alpha_{k,0}$). Pick a non-zero $c_k \in \mathbb{R}$ and set

$$f_{k+1} := f_k^{p_k} - c_k \prod_{j=0}^{k-1} f_j^{\alpha_{k,j}}.$$

Set ω_{k+1} to be a rational number less than $p_k\omega_k$.

Construction of S from a finite sequence of f_k's: Assume f_k's and ω_k's have been constructed up to some $l \geq 1$. Construct p_l and $\alpha_{l,0}, \ldots, \alpha_{l,l-1}$ as in the inductive step. Define

$$f_{l+1,i} := f_l^{p_l} - c_{l,i} \prod_{k=0}^{l-1} f_k^{\alpha_{l,k}},$$

where $c_{l,1}, c_{l,2}$ are distinct real numbers such that each $f_{l+1,i}$ defines a curve C_i on $\mathbb{R}^2 \setminus y$-axis. Let $C_i^{\mathbb{C}}$ be the curve defined by $f_{l+1,i}$ in $\mathbb{C}^2 \setminus y$-axis. Then each $C_i^{\mathbb{C}}$ has a unique irreducible branch for which $|x| \to \infty$. It follows that either C_i has a unique branch for which $x \to \infty$, or it has two such branches which come from the same irreducible branch of $C_i^{\mathbb{C}}$. In any event, there is a unique 'top' branch of C_i for which $x \to \infty$; let us denote it by C_i^{top}. Pick $r > 0$ and let S be the region to the right of $x = r$ and bounded by C_1^{top} and C_2^{top}.

The following is an immediate corollary of the results of the preceding subsection and the observations that $f_0, \dots, f_l$ are precisely the key-forms of δ_S^* and $\omega_l = \delta_S^*(f_l)$. Note that the result does not change if we took S to be the region bounded by the 'bottom' branches of C_i, or if we took corresponding branches of C_i for which $x \to -\infty$.

Corollary 4.22.

(1) $\mathcal{B}_0(S) = \mathbb{R}$ *iff one of the following is true:*
   (a) $\omega_l > 0$, *or*
   (b) $\omega_l = 0$ *and* $f_l \notin \mathbb{R}[x, y]$ *(equivalently,* $\omega_l = 0$ *and* $\alpha_{k,0} < 0$ *for some* k, $1 \le k \le l - 1$*).*
(2) $\mathcal{B}(S)$ *is a finitely generated algebra over* $\mathbb{R}$ *iff one of the following conditions hold:*
   (a) $\omega_l < 0$, *or*
   (b) $\omega_l \ge 0$ *and* $f_k \in \mathbb{R}[x, y]$ *for* $1 \le k \le l$ *(equivalently,* $\omega_l \ge 0$ *and* $\alpha_{k,0} \ge 0$ *for* $1 \le k \le l - 1$*).*
(3) *There exists* $d > 0$ *such that* $\mathcal{B}_d(S)$ *is* not *a finite dimensional module over* $\mathcal{B}_0(S)$ *iff both of the following conditions hold:*
   (a) $\omega_l = 0$ *and*
   (b) $f_l \notin \mathbb{R}[x, y]$ *(or equivalently,* $\alpha_{k,0} < 0$ *for some* k, $1 \le k \le l - 1$*).*

Example 4.23. Consider a sequence of key-forms starting with $x, y, y^2 - x^5$ (which corresponds to choices $\omega_1 := 5/2$ and $c_1 := 1$).

(1) Take $\omega_2 := 1$, $l := 2$, $c_{2,1} := 0$ and $c_{2,2} := 1$. Let S be the region defined by $x \ge 1$, $y \ge 0$, $x \ge y^2 - x^5 \ge 0$. Then $\mathcal{B}_0(S) = \mathbb{R}$ and $\mathcal{B}(S)$ is a finitely generated algebra over $\mathbb{R}$.
(2) Take $\omega_2 := 0$, $l := 2$, $c_{2,1} := 0$ and $c_{2,2} := 1$. Let S be the region defined by $x \ge 1$, $y \ge 0$, $1 \ge y^2 - x^5 \ge 0$. Then $\mathcal{B}_0(S) \supsetneq \mathbb{R}$ and $\mathcal{B}(S)$ is a finitely generated algebra over $\mathbb{R}$.
(3) Take $\omega_2 := 3/2$, $c_2 := 1$, $\omega_3 := 1$, $l := 3$, $c_{3,1} := 0$ and $c_{3,2} := 1$. Let S be the region defined by $x \ge 1$, $y \ge 0$, $x \ge y^2 - x^5 - yx^{-1} \ge 0$. Then $\mathcal{B}_0(S) = \mathbb{R}$ and $\mathcal{B}(S)$ is not a finitely generated algebra over $\mathbb{R}$. But $\mathcal{B}_d(S)$ is finite dimensional over $\mathbb{R}$ for all d.

(4) Take $\omega_2 := 3/2$, $c_2 := 1$, $\omega_3 := 0$, $l := 3$, $c_{3,1} := 0$ and $c_{3,2} := 1$. Let S be the region defined by $x \geq 1$, $y \geq 0$, $1 \geq y^2 - x^5 - yx^{-1} \geq 0$. Then $\mathcal{B}_0(S) = \mathbb{R}$ and $\mathcal{B}(S)$ is not a finitely generated algebra over $\mathbb{R}$. Moreover, there exists $d > 0$ such that $\mathcal{B}_d(S)$ is infinite dimensional over $\mathbb{R}$.

4.4.4. Two tentacles which behave like one

In this subsection we assume that

(1) ξ is an indeterminate,
(2) $\phi(x) = \sum_{j=1}^{k} a_j x^{m_j/2} \in \mathbb{R}[x^{1/2}, x^{-1/2}]$ for some positive integer k and integers $m_1 > m_2 > \cdots > m_k$ such that $\gcd(m_1, \ldots, m_k) = 1$, and
(3) $\omega \in \mathbb{Q}$, $\omega < m_k/2 = \operatorname{ord}_x(\phi)$.

Define

$$\phi_1(x) := \sum_{j=1}^{k} a_j x^{m_j} + \xi x^{2\omega},$$

$$\phi_2(x) := \sum_{j=1}^{k} (-1)^{m_j} a_j x^{m_j} + \xi x^{2\omega}.$$

Proposition 4.24. *Let $\tilde{S}, S_1, S_2$ be semialgebraic subsets of $\mathbb{R}^2$ which satisfy assumptions (A1) and (A2). Assume that the generic degree-wise Puiseux series (see Definition 4.4) corresponding to $\tilde{S}, S_1, S_2$ is respectively $\phi(x) + \xi x^{\omega}$, $\phi_1(x) + \xi x^{2\omega}$, and $\phi_2(x) + \xi x^{2\omega}$. Set $S = S_1 \cup S_2$. Then $\mathcal{B}(S)$ is integral over $\mathcal{B}(\tilde{S})$. In particular, for each of Questions 1.1–1.4, its answer is positive for S iff it is positive for $\tilde{S}$.*

Proof. Consider the map $(x, y) \mapsto (x^2, y)$. Then it is not hard to see that $\delta^*_{S_1}$ and $\delta^*_{S_2}$ extend $\delta^*_{\tilde{S}}$ via the pull-back by f. [12, Lemma A.3] then shows that $\mathcal{B}(S)$ is the integral closure of $f^*\mathcal{B}(\tilde{S})$. The proposition follows immediately. □

Example 4.25 *(The example of Section 3 revisited).* The generic degree-wise Puiseux series corresponding to S_1, S_2 of Section 3 are respectively

$$\tilde{\phi}_1(x, \xi) := -x^3 + x^{-2} + \xi x^{-3},$$
$$\tilde{\phi}_2(x, \xi) := x^3 + x^{-2} + \xi x^{-3}.$$

Let $\tilde{S}$ be a tentacle with generic degree-wise Puiseux series $\tilde{\phi}(x, \xi) := x^{3/2} + x^{-1} + \xi x^{-3/2}$; we may construct such $\tilde{S}$ using the procedure in Section 4.4.3; e.g. take $\tilde{S}$ to be the set defined by $x \geq 1$, $y \geq 0$, $1 \geq y^2 - x^3 - 2yx^{-1} \geq 0$. Then it follows exactly as in the last case of Example 4.23 that $\mathcal{B}_0(\tilde{S}) = \mathbb{R}$ and there exists $d > 0$ such $\mathcal{B}_d(\tilde{S})$ is infinite dimensional over $\mathbb{R}$. Proposition 4.24 therefore implies that the same is true for $S := S_1 \cup S_2$, as it was indeed shown in Section 3.

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References

[1] S.S. Abhyankar, T.T. Moh, Embeddings of the line in the plane, J. Reine Angew. Math. 276 (1975) 148–166.
[2] J. Cimprič, M. Marshall, T. Netzer, Closures of quadratic modules, Israel J. Math. 183 (1) (2011) 445–474.
[3] S. Friedland, J. Milnor, Dynamical properties of plane polynomial automorphisms, Ergodic Theory Dynam. Systems 9 (1989).
[4] E.K. Haviland, On the moment problem for distribution functions in more than one dimension II, Amer. J. Math. 58 (1936) 164–168.
[5] S. Krug, Geometric interpretations of a counterexample to Hilbert's 14th problem and rings of bounded polynomials on semialgebraic sets, Preprint, 2011.
[6] S. Kuhlmann, M. Marshall, Positivity, sums of squares and the multi-dimensional moment problem, Trans. Amer. Math. Soc. 354 (11) (2002) 4285–4301.
[7] R. Lazarsfeld, Lectures on linear series, in: Complex Algebraic Geometry, Park City, UT, 1993, in: IAS/Park City Math. Ser., vol. 3, 1997, pp. 161–219.
[8] S. MacLane, A construction for absolute values in polynomial rings, Trans. Amer. Math. Soc. 40 (3) (1936) 363–395.
[9] M. Marshall, Positive Polynomials and Sums of Squares, Math. Surveys Monogr., vol. 146, American Mathematical Society, Providence, RI, 2008.
[10] P. Mondal, An effective criterion for algebraic contractibility of rational curves, Preprint, 2013.
[11] P. Mondal, How to determine the sign of a valuation on $\mathbb{C}[x, y]$?, Preprint, 2013.
[12] P. Mondal, Is the intersection of two finitely generated subalgebras of a polynomial ring also finitely generated?, Preprint, 2013.
[13] P. Mondal, Projective completions of affine varieties via degree-like functions, Preprint, 2013.
[14] T. Netzer, Stability of quadratic modules, Manuscripta Math. 129 (2) (2009) 251–271.
[15] T. Netzer, An elementary proof of Schmüdgen's theorem on the moment problem of closed semialgebraic sets, Proc. Amer. Math. Soc. 136 (2008) 529–537.
[16] D. Plaumann, Bounded polynomials, sums of squares and the moment problem, PhD thesis, University of Konstanz Konstanz Online Publication System, 2008.
[17] D. Plaumann, C. Scheiderer, The ring of bounded polynomials on a semi-algebraic set, Trans. Amer. Math. Soc. 364 (2012) 4663–4682.
[18] V. Powers, C. Scheiderer, The moment problem for non-compact semialgebraic sets, Adv. Geom. 1 (2001) 71–88.
[19] C. Scheiderer, Non-existence of degree bounds for weighted sums of squares representations, J. Complexity 21 (2005) 823–844.
[20] K. Schmüdgen, The K-moment problem for compact semi-algebraic sets, Math. Ann. 289 (1991) 203–206.
[21] K. Schmüdgen, On the moment problem of closed semi-algebraic sets, J. Reine Angew. Math. 558 (2003) 225–234.
[22] M. Suzuki, Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace $\mathbf{C}^2$, J. Math. Soc. Japan 26 (1974) 241–257.
[23] L. Szpiro, Fonctions d'ordre et valuations, Bull. Soc. Math. France 94 (1966) 301–311.
[24] C. Vinzant, Real radical initial ideals, J. Algebra 352 (1) (2012).